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# Numerical solutions for nonlinear elliptic problems based on first-order system

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ABSTRACT

We present a theoretical analysis of numerical solutions for nonlinear elliptic problems. Our analysis is based on the abstract approximation theory for branches of nonsingular solutions developed by Brezzi, Rappaz, and Raviart (BRR). In most cases of finite element analysis, the same spaces are used for the domain and range of the nonlinear operator concerning the application of BRR theory. This results in a loss of accuracy in the error estimates while numerical experiments show optimal convergence rates. The main contribution of this paper is a theoretical analysis for the optimal convergence rates. This is achieved by choosing different spaces for the domain and range of the nonlinear operator used in the application of BRR theory. Previously, this idea was used for an analysis of Petrov–Galerkin formulation for the BRR theory. Our analysis is developed for the first-order least-squares finite element methods and the mixed Galerkin methods based on first-order systems of equations.



## 1. Introduction

We consider numerical solutions for second-order nonlinear elliptic problems with gradient nonlinearities. The purpose of this paper is twofold. The first contribution is making an observation that one can use different spaces for the domain and range of nonlinear operator in the application of the abstract theory developed by Brezzi, Rappaz, and Raviart (BRR) [1,2]. In most cases in the context of finite element analysis, the same spaces are chosen for the domain and range of the nonlinear operator. This results in a loss of accuracy in the error estimates while numerical experiments show optimal convergence rates, e.g. see [3]. A novel feature of our work is choosing different spaces for the domain and range of the nonlinear operator. This turns out to be an important ingredient to obtain error estimates for numerical methods for nonlinear problems. As a result, the underlying mesh does not need to be quasi-uniform and optimal error estimates are obtained with respect to the finite element spaces under a mild mesh condition. Previously, the idea of choosing different spaces for the domain and range of the nonlinear operator is used in [4,5]. The authors obtained error estimates for Petrov–Galerkin methods to approximate nonlinear elliptic problems.

The second contribution is that we use the above observation to obtain optimal error estimates for least-squares (LS) and mixed Galerkin finite element methods to solve second-order nonlinear elliptic problems. Efforts to use LS methodology to solve nonlinear problems are mostly confined to the Navier–Stokes equations. In this paper, we develop a LS method to solve second-order scalar elliptic nonlinear problems. The same problems are solved by the mixed Galerkin methods in [3]. LS approaches have advantages over the other numerical methods. For example, it does not have to satisfy the inf–sup condition required by the mixed methods and the resulting algebraic equations involve symmetric and positive definite

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matrices. Also, it has a built-in a posteriori error estimators, which can be used in the iterative procedures for solving nonlinear problems. We refer the reader to [13–19] and references therein for LS methods based on a first-order system. In [3], abstract approximation theory for branches of nonsingular solutions developed by BRR was used for an analysis of the mixed Galerkin methods. However, the error estimate is not optimal while numerical experiments show optimal convergence, see [3, Section 6]. We provide optimal error estimates for both LS approach developed in this paper and for the mixed Galerkin finite element methods. Familiarity with [3] is helpful since our analysis will be using some of the results developed there.

The paper is organized as follows. Section 2 introduces mathematical equations for the second-order scalar elliptic partial differential equations and presents basic assumptions for the equations. In Section 3, finite element spaces are introduced and Ritz projection is presented along with a condition for local mesh function for optimal error estimates on shape regular meshes. In Section 4, abstract theory for solving nonlinear problems is introduced. In Section 5, we consider LS finite element approximation for elliptic problems with gradient nonlinearities. In Section 6, optimal error estimate using the mixed Galerkin methods is presented.

## 2. Problem formulation

Let $H^s(\Omega)$ denote the Sobolev space of order $s$ defined on $\Omega$. The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. For $s = 0$, $H^s(\Omega)$ coincides with $L^2(\Omega)$. We shall use the spaces

$$V = H_0^1(\Omega) \quad \text{and} \quad \mathbf{W} = H(\text{div}),$$

with norms $\|u\|_1^2 = (u, u) + (\nabla u, \nabla u)$, $\|\boldsymbol{\sigma}\|_{H(\text{div})}^2 = (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\sigma}) + (\boldsymbol{\sigma}, \boldsymbol{\sigma})$.

### 2.1. Model problems and basic assumptions

Our model problem is

$$\begin{cases} -\nabla \cdot \mathcal{A}\nabla u + b(x, u, \nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega$ is a bounded domain in $R^n$, $n = 2, 3$, with a smooth boundary $\partial\Omega$ and $\mathcal{A}(x)$ is uniformly symmetric and positive definite matrix. We shall assume that (2.1) has a unique solution $u \in H^s(\Omega)$, where $s = 2 + \epsilon$ for $n = 2$, and $s = \frac{5}{2} + \epsilon$ for $n = 3$ with $0 < \epsilon < 1$.

We present some conditions concerning the function $b : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. These conditions are the same ones presented in [3] and we refer the readers to [3] for more detailed discussions and references therein.

(C1) $b(x, y, \mathbf{z})$ is of $C^{1,\alpha}$-class, $0 < \alpha \le 1$, with respect to $y \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^n$: there exists a positive constant $C$ independent of $x, y, \mathbf{z}$ such that

$$|b_\omega(x, y, \mathbf{z}) - b_\omega(x, \bar{y}, \bar{\mathbf{z}})| \le C(|y - \bar{y}|^\alpha + |\mathbf{z} - \bar{\mathbf{z}}|^\alpha)$$

for any $(x, y, \mathbf{z}), (x, \bar{y}, \bar{\mathbf{z}}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Here, $b_\omega$ denotes $b_y$ or $b_{z_i}$, $i = 1, \ldots, n$.

(C2) The function $b$ and its partial derivatives $b_y$ and $b_{\mathbf{z}}$ satisfy the following growth conditions: for some constant $C$

$$|b(x, y, \mathbf{z})| \le C(1 + |y|^\beta + |\mathbf{z}|^\beta),$$
$$|b_y(x, y, \mathbf{z})| \le C(1 + |y|^{\beta-1} + |\mathbf{z}|^{\beta-1}),$$

and

$$|b_{\mathbf{z}}(x, y, \mathbf{z})| \le C(1 + |y|^{\beta-1} + |\mathbf{z}|^{\beta-1}),$$

where $\beta \ge \alpha + 1$. Or for some constant $C$

$$|b(x, y, \mathbf{z})| \le C(1 + |y|^\gamma + |\mathbf{z}|^\beta),$$
$$|b_y(x, y, \mathbf{z})| \le C(1 + |y|^{\gamma-1}),$$

and

$$|b_{\mathbf{z}}(x, y, \mathbf{z})| \le C(1 + |\mathbf{z}|^{\beta-1}),$$

where $\gamma > 1$ and $\beta \ge \alpha + 1$.

**Remark 2.1.** As discussed in [3], our model problem is a special case of Hamilton–Jacobi–Bellman type equation arising, for example, in optimal stochastic control or population dynamics.

By introducing a flux variable $\boldsymbol{\sigma} = -\mathcal{A}\nabla u$, (2.1) is equivalent to finding $(u, \boldsymbol{\sigma}) \in V \times \mathbf{W}$ such that

$$\begin{cases} \boldsymbol{\sigma} + \mathcal{A}\nabla u = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{\sigma} + c(u, \boldsymbol{\sigma}) = 0 & \text{in } \Omega, \end{cases} \tag{2.2}$$

where $c(u, \boldsymbol{\sigma}) = b(u, -\mathcal{A}^{-1}\boldsymbol{\sigma})$.