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On convergence and inherent oscillations within computational methods employing fictitious sources

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ABSTRACT

Recent studies of the method of auxiliary sources (MAS) and the method of fundamental solutions (MFS) have shown that it is possible to have convergence of the field, which is the finally desired quantity, together with divergence of the fictitious sources, which are intermediate quantities. The aforementioned studies concern a simple scattering problem, where the scattered field satisfies the Helmholtz equation and the fictitious sources are placed symmetrically. In the present paper, we extend the previous results in two directions, one having to do with the geometry and the other with frequency. More precisely, we first show that the above convergence/divergence phenomena can also occur when the fictitious sources are placed asymmetrically and then study the nature of this divergence in detail. Second, we show that all findings carry over to the case of Laplace's equation. We, additionally, point out certain differences of the discussed phenomena with the well-known ones of internal resonances.

1. Introduction

The key ideas behind the methods of fundamental solutions (MFS) [1,2], of auxiliary sources (MAS) [3,4], of fictitious sources (MFS) [5,6], and the source model technique (SMT) [7,8] were introduced in [9,10] and concern the representation of the approximate solution of a boundary value problem as a finite superposition of fields of fictitious (auxiliary) sources located outside the problem's domain. Each of these fields is proportional to the fundamental solution of the governing partial differential equation while the fields' amplitudes are determined by the boundary conditions on the actual physical boundary. Extended reference lists concerning these methods as well as their efficient implementation in the modeling of wave phenomena arising in diverse application areas are included in [2,4,11,12]. In the sequel, we will use the terminology of MAS to be compatible with and show the extensions from previous works. Still, we note that our findings can be extended to the aforementioned computational methods employing fictitious sources.

For the simplest case, that of scattering by an externally illuminated perfect electric conductor (PEC), the basic steps of MAS are as follows: (i) Assume that the PEC scatterer is absent. (ii) Determine N fictitious electric currents located on an auxiliary surface placed inside the space originally occupied by the scatterer; this is done via an $N \times N$ system of linear algebraic equations requiring cancellation of the total field ($=$ incident field $+$ field due to the N sources) on N collocation points located on the outer surface of the space originally occupied by the PEC scatterer. (iii) With the N auxiliary currents thus determined, find the field due to them outside the PEC; this field, which we will call "MAS field", is an approximation to the desired scattered field.

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The auxiliary surface is usually smooth and closed, and we can ideally obtain the correct scattered field by completely filling this surface with currents, i.e., in the limit $N \to \infty$. Furthermore, as N grows it is advantageous – for purposes of stability – to obtain the correct field with only small changes in the "normalized MAS currents" obtained by dividing each current by the arc-length distance to its adjacent current. In other words, it is desirable that the normalized discrete MAS currents converge, as $N \to \infty$, to a well-defined, continuous, and smooth surface current density.

Exterior fields due to such surface current densities are analytic functions of the space variables; it is thus readily understood that the above-described convergence of the normalized MAS currents cannot occur if the analytic continuation of the scattered field to points interior to the PEC surface presents a singularity that is exterior to the auxiliary surface. As discussed in [13–15], there are two possible sources for such singularities, the incident field and the shape of the PEC surface.

Surprisingly, it is possible for the MAS *field* to converge to the correct scattered field even in the case (described above) where the MAS *currents* diverge. This phenomenon was first (to the best of our knowledge) demonstrated in [16], and then further discussed in [17]: For the problem of scattering by a 2-D circular PEC cylinder illuminated by an infinitely long line source, these two references show that the phenomenon occurs when the radius of the auxiliary surface (which is also a circular cylinder) is smaller than a certain critical radius. Due to the simplicity of the problem, this radius can be explicitly calculated and turns out to be the same with the radius of the image in magnetostatics [16]. The coincidence of the singularity in the present (Helmholtz) case and that of the image in the Laplace case is to be expected from the general results of [13] (see also [18]). Since the singularity position is known in advance, we know *a priori* whether divergence will occur or not (something generally not known in more complicated scattering problems).

When N is large enough in the aforementioned circular problem, the most telling numerical sign of the divergence of the MAS currents is the appearance of unphysical oscillations [16,17]. Therefore, the phenomenon under discussion is peculiar in the sense that one obtains the correct final result (namely, the scattered field) from an unnatural, rapidly oscillating intermediate result (namely, the MAS currents). For the simple scattering problem, the convergence of the field is shown in [16] analytically. Such an analytical demonstration is important because it is well-known (see, e.g., [4,19,20]) that MAS is plagued by numerical difficulties, which can greatly corrupt computer-generated MAS fields and mask their true behavior. A further analytical result is a large-N asymptotic formula for the oscillating and divergent MAS currents [17], which well agrees with numerical results in cases where these are not corrupted by roundoff and matrix ill-conditioning. This formula shows that the oscillations are exponentially large in the parameter N, clearly reveals the oscillations' origin, and distinguishes them from similar – but unconnected – oscillations that can occur because of matrix ill-conditioning; see, for example, the numerical results in [21]. While the phenomenon can be related to antenna superdirectivity [22] (see also [23]), it is (as we will stress in the present paper) unrelated to the much-studied effect of internal (cavity) resonances. It is also noteworthy that no similar phenomenon occurs when the same simple scattering problem is solved via a discretization of the Extended Integral Equation (EIE) [21]. This is logical because, as remarked in [24], the analytic continuation of the total exterior field does not vanish in the interior (otherwise it would vanish everywhere) [21].

The above-described phenomenon is a difficulty within the MAS. Since it occurs in a simple scattering problem (like the circular problem in [16,17]), it must also occur in more complicated problems. This is indeed the case: see [25] for an interior illuminating line source (certain aspects are also discussed in [15]), and [26] for a dielectric scatterer. Purely numerical results suggest that a similar phenomenon (i.e., convergence of the MAS field generated by divergent and oscillatory MAS currents) also occurs in the case of a 2-D elliptical PEC scatterer [17]. Results concerning a certain combined MAS-EIE method are contained in [27]; this reference and [25] also contain detailed discussions on the practical consequences of the phenomenon and its relation to instability.

The purpose of this paper is to extend the results of [16,17] in two directions, one (Section 2) having to do with the geometry and the other (Section 3) with frequency. In all circular problems discussed previously, and in the elliptical problem as well, the N MAS source points exhibit the same sort of symmetries as do the N collocation points: For example, each of the two sets of points is symmetric with respect to both the x- and y-axes (in the elliptical case, these axes respectively coincide with the ellipse's major and minor axes). The primary purposes of Section 2 are: (i) to show that these symmetry aspects are irrelevant to the occurrence of oscillations; (ii) to show, more importantly, that they are also irrelevant to the concurrent convergence of the MAS field to the true field; and (iii) to point out a number of features particular to asymmetric cases. This is done by applying the MAS to a circular scattering problem with sources and collocation points that are offset from the symmetry axes: we analytically show that the MAS field converges to the true field, and we extend the aforementioned large-N asymptotic formula of [17]. Our approach includes a detailed study of effects due to changes in the parameters and avoids confusion with similar effects due to roundoff. Section 2 additionally points out certain differences with the well-known phenomenon of internal resonances and sheds further light on the relevance of the analytic continuation of the scattered field to points interior to the scatterer; issues related to this analytic continuation have caused controversy, see [25,28,29], as well as additional relevant references in [25].

Various engineering applications, including e.g. crack singularities [30] and inductance calculations [31,32], often involve two-dimensional potential, magnetostatic or quasi-static problems solved numerically using Laplace's equation. The purpose of Section 3 is to extend the results of Section 2 to the corresponding Laplace problem (we specifically use the language of magnetostatics). The asymptotic formula of Section 2, and the one in [17] as well, is independent of the wavenumber k—only ratios of the various distances appear. This leads one to suspect that the asymptotic formula continues to hold in the magnetostatic case of zero frequency. But because the Green's functions in the Laplace and Helmholtz cases are different, the existing derivation (of Section 2) does not easily bring this out. The conclusion of Section 3 is concisely arrived