Computers and Mathematics with Applications 69 (2015) 688–695

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Two-phase porous media flows with dynamic capillary effects and hysteresis: Uniqueness of weak solutions

X. Cao [a,*], I.S. Pop [a,b]

[a] CASA, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

[b] Institute of Mathematics, Johannes Bruns GT. 12, University of Bergen, Bergen, Norway

ARTICLE INFO

Article history:
Received 24 October 2014
Received in revised form 22 January 2015
Accepted 8 February 2015
Available online 27 February 2015

Keywords:
Dynamic capillary pressure
Two-phase flow
Hysteresis
Weak solution
Uniqueness

ABSTRACT

In this paper, we obtain the uniqueness of weak solutions for a two phase flow model in a porous medium. A particularity of the model is that the dynamic effects and hysteresis are included in the capillary pressure. To prove the uniqueness, we define an auxiliary elliptic system.

1. Introduction

We consider a mathematical model for two-phase flow in a porous medium. Two immiscible fluid phases (for example, water — the wetting phase, and oil — the non-wetting one) are flowing through a porous medium occupying a bounded, connected domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$. $\bar{\Omega}$ and $\partial\Omega$ denote the closure, respectively boundary of Ω. We let $t \in (0, T]$ be the time variable, where $T > 0$ is a given maximal time. The phase pressures are denoted by p_w, p_n. The non-wetting phase saturation is s. We assume the porous medium is saturated by the two phases, so no other flowing phase is present. This means, the wetting phase saturation is $1 - s$. Then from the Darcy law and mass conservation for each fluid, one obtains (see [1,2])

$$\partial_t s - \nabla \cdot (k_n(s)\nabla p_n) - \nabla \cdot (k_n(s)\overrightarrow{g}) = 0, \tag{1}$$

$$-\partial_t s - \nabla \cdot (k_w(s)\nabla p_w) - \nabla \cdot (k_w(s)\overrightarrow{g}) = 0. \tag{2}$$

Here $\overrightarrow{g} \in \mathbb{R}^d$ is the gravity vector in direction $-\overrightarrow{e_d} = (0, \ldots, 0, -1) \in \mathbb{R}^d$. $k_n(s)$, $k_w(s)$ are the relative permeabilities, two nonlinear functions depending on s. The system is closed by the relation between the phase pressures and saturation. Standardly, one assumes that

$$p_n - p_w = p_c(s),$$

p_c being a given increasing function in s (see [3]). These are so-called equilibrium models. However, experiments [4–6] have invalidated such models. In particular, one should distinguish between infiltration, when the wetting phase is displacing

* Corresponding author. Tel.: +31 40 2475546.
E-mail addresses: x.cao@tue.nl (X. Cao), i.pop@tue.nl (I.S. Pop).
URL: http://www.win.tue.nl/casa/people/tempstaff/168.html (X. Cao).

http://dx.doi.org/10.1016/j.camwa.2015.02.009
0898-1221/

the non-wetting phase, and the opposite process, drainage. The switch between the two situations above is achieved in so-called hysteresis models. Further, experiments for which classical, equilibrium models would predict monotone infiltration saturation profiles, actually revealed non-monotone profiles (saturation overshoot) (see [7]). Such profiles are instead allowed in dynamic capillarity models. Both effects mentioned above are included in:

$$p_n - p_w \in p_c(s) + \gamma(x)\mathrm{sign}(\partial_t s) + \tau\partial_t s. \tag{3}$$

The second term on the right hand side models play-type hysteresis (see [8,9]), while the last one accounts for dynamic capillarity (see [10]). In (3), $\gamma \geq 0$, $\tau > 0$ are given, while sign is the multi-valued graph:

$$\mathrm{sign}(\xi) = \begin{cases} 1 & \text{if } \xi > 0, \\ -1 & \text{if } \xi < 0, \\ [-1, 1] & \text{if } \xi = 0. \end{cases} \tag{4}$$

Following [11], we also consider the Lipschitz continuous function $\Psi : \mathbb{R} \times \Omega \to \mathbb{R}$

$$\Psi(\xi, x) = \begin{cases} \dfrac{\xi - \gamma(x)}{\tau} & \text{for } \xi > \gamma(x), \\ 0 & \text{for } \xi \in [-\gamma(x), \gamma(x)], \\ \dfrac{\xi + \gamma(x)}{\tau} & \text{for } \xi < -\gamma(x). \end{cases} \tag{5}$$

Clearly, for a. e. $\xi \in \mathbb{R}$, one has

$$0 \leq \partial_\xi \Psi(\xi, x) \leq 1/\tau, \tag{6}$$

With this, (3) rewrites

$$\partial_t s = \Psi(p_n - p_w - p_c(s), x). \tag{7}$$

The model (1), (2), (7) is complemented by initial and boundary conditions:

$$s(0, \cdot) = s_0, \tag{8}$$

$$p_n = p_w = 0 \quad \text{at } \partial\Omega, \text{ for all } t \geq 0. \tag{9}$$

Remark 1. Other boundary conditions are possible (see Remark 3), but for clarity, we restrict the presentation to (9).

We make the following assumptions:

A1: The functions $k_w, k_n : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. Further, $\delta, M_k > 0$ exist, such that $\delta \leq k_n(s), k_w(s) \leq M_k < +\infty$, for all $s \in \mathbb{R}$.
A2: $p_c(\cdot) \in C^1(\mathbb{R})$ is increasing and Lipschitz continuous, there exist $m_p, M_p > 0$, such that $m_p \leq \|p_c\|_{\mathrm{Lip}} \leq M_p < +\infty$, for all $s \in \mathbb{R}$.
A3: Ω is a $C^{1,\beta}$ domain with $0 < \beta \leq 1$.
A4: $\gamma(x) \in C^{0,1}(\bar{\Omega})$.
A5: $s_0 \in C^{0,\beta}(\bar{\Omega})$.

Remark 2. Commonly, the permeabilities and capillary pressure model encountered in the literature [12,13] are

$$k_n(s) = s^p, \qquad k_w(s) = (1-s)^q, \quad \text{with } p, q > 1,$$

and

$$p_c(s) = (1-s)^{-\frac{1}{\mu}}, \quad \mu > 1, \text{ for } s \in [0, 1].$$

Then (**A1**) and (**A2**) are not satisfied when s approaches 0 or 1. We consider here a regularized approximation of these functions.

Below we use standard notation in the theory of partial differential equations. For any $h \in C^{0,\beta} : \Omega \to \mathbb{R}$, we define the norm

$$\|h\|_{C(\bar{\Omega})} := \sup_{x \in \Omega} |h(x)|,$$

the βth-Hölder semi-norm of h

$$[h]_{C^{0,\beta}(\bar{\Omega})} := \sup_{x,y \in \Omega, x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\beta},$$

and the βth-Hölder norm of h

$$\|h\|_{C^{0,\beta}(\bar{\Omega})} := \|h\|_{C(\bar{\Omega})} + [h]_{C^{0,\beta}(\bar{\Omega})}.$$