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Some generalized Ostrowski–Grüss type integral inequalities

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ABSTRACT

In this paper, we establish some new Ostrowski–Grüss type integral inequalities involving $(k-1)$ interior points in 1D case, which are generalizations of some known results in the literature, and one of which is sharp. Then we deduce an Ostrowski–Grüss type integral inequality in 2D case involving $(k-1)^2$ interior points for the first time. We also present one application on the estimate of error bound for numerical integration formula, in which a sharp error bound for a new numerical integration formula is provided by the results established.

1. Introduction

In recent years, the research for the Ostrowski type inequalities has been a hot topic in the literature. The Ostrowski type inequality, which can be used to estimate the absolute deviation of a function from its integral mean, was originally presented by Ostrowski in [1] as follows (see also in [2, pp. 468]).

Theorem 1.1. *Let $f : I \to R$ be a differentiable mapping in the interior Int I of I, where $I \subset R$ is an interval, and let $a, b \in \operatorname{Int} I$. $a < b$. If $|f'(t)| \le M$, $\forall t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M, \quad \text{for } x \in [a, b].$$

Since then, various generalizations of the Ostrowski inequality have been established (for example, see [3–15] and the references therein), one of which is the inequalities of Ostrowski–Grüss type (for example, see [16–23]). Such inequalities can be used to provide explicit error bounds for some known and some new numerical quadrature formulas. The first inequality of the Ostrowski–Grüss type was presented by Dragomir and Wang in [16], which reads as in the following theorem.

Theorem 1.2 ([16, Theorem 2.1]). *Let $I \subset R$ be an open interval, $a, b \in I$, $a < b$. If $f : I \to R$ is a differential function such that there exist constants $\gamma, \Gamma \in R$ with $\gamma \le f'(x) \le \Gamma$, $x \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma - \gamma) \quad \text{for all } x \in [a, b]. \tag{1}$$

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In [17], Cheng presented the sharp version for (1), which is shown in the following theorem.

Theorem 1.3 *([17, Theorem 1.5]). Let the assumptions of Theorem 1.1 hold. Then for all $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)(\Gamma - \gamma). \tag{2}$$

The inequality (2) is sharp in the sense that the constant $\frac{1}{8}$ cannot be replaced by a smaller one.

Motivated by the above work, in this paper, we will present some new Ostrowski–Grüss type integral inequalities, which are generalizations of Theorems 1.2 and 1.3 to the case involving $(k-1)$ interior points $x_i,\ i = 1, 2, \ldots, k-1$, and then extend Theorem 1.2 to 2D case involving $(k-1)^2$ interior points $(x_i, y_j),\ i, j = 1, 2, \ldots, k-1$. We will also present one application for the results established, in which sharp error bound for a new numerical integration formula is derived under some suitable conditions.

2. Main results

Lemma 2.1 (*Montgomery Identity*). *Let $I \subset R$ be an open interval, $a, b \in I$, $a < b$. $f : I \to R$ is a differential function such that there exist constants $\gamma, \Gamma \in R$ with $\gamma \le f'(x) \le \Gamma$, $x \in [a, b]$. Furthermore, suppose that $x_i \in [a, b],\ i = 0, 1, \ldots, k$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a, b]$, and $m_i \in [x_{i-1}, x_i],\ i = 1, 2, \ldots, k$, $m_0 = a,\ m_{k+1} = b$. Then we have*

$$\sum_{i=0}^{k}(m_{i+1} - m_i)f(x_i) = \int_a^b f(t)dt + \int_a^b k(t, I_k)f'(t)dt, \tag{3}$$

where

$$k(t, I_k) = \begin{cases} t - m_1, & t \in [x_0, x_1), \\ t - m_2, & t \in [x_1, x_2), \\ \ldots \\ t - m_{k-1}, & t \in [x_{k-2}, x_{k-1}), \\ t - m_k, & t \in [x_{k-1}, x_k]. \end{cases} \tag{4}$$

Lemma 2.1 can be obtained by taking $\mathbb{T} = R$ in [24, Lemma 1].

Lemma 2.2 *([25, pp. 295, Grüss' Inequality]). Let $f, g : [a, b] \to R$ be two integrable functions such that $\phi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are constants. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt\frac{1}{b-a}\int_a^b g(t)dt\right| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{5}$$

Theorem 2.3. *Under the conditions of Lemma 2.1, we have the following inequality*

$$\left|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1} - m_i)f(x_i) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{(b-a)^2}\left[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1} - x_i)\right]\right|$$
$$\le \frac{1}{4}(b-a)(\Gamma - \gamma). \tag{6}$$

Proof. From (4), we can obtain $t - b \le k(t, I_k) \le t - a$. So a combination of Lemmas 2.1 and 2.2 yields

$$\left|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1} - m_i)f(x_i) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{b-a}\int_a^b k(t, I_k)dt\frac{1}{b-a}\int_a^b f'(t)dt\right|$$
$$\le \frac{1}{4}[(t-a) - (t-b)](\Gamma - \gamma) = \frac{1}{4}(b-a)(\Gamma - \gamma). \tag{7}$$

On the other hand,

$$\int_a^b k(t, I_k)dt = \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(t - m_{i+1})dt = \sum_{i=0}^{k-1}\left[\frac{(x_{i+1} - m_{i+1})^2}{2} - \frac{(x_i - m_{i+1})^2}{2}\right]$$