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# Delayed feedback on the 3-D chaotic system only with two stable node-foci

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ABSTRACT

In this paper, we investigate the effect of delayed feedbacks on the 3-D chaotic system only with two stable node-foci by Yang et al. The stability of equilibria and the existence of Hopf bifurcations are considered. The explicit formulas determining the direction, stability and period of the bifurcating periodic solutions are obtained by employing the normal form theory and the center manifold theorem. Numerical simulations and experimental results are given to verify the theoretical analysis. Hopf bifurcation analysis can explain and predict the periodic orbit in the chaotic system with direct time delay feedback. We also find that the control law can be applied to the chaotic system only with two stable node-foci for the purpose of control and anti-control of chaos. Finally, some numerical simulations are given to illustrate the effectiveness of the results found.



## 1. Introduction

In 1963, Lorenz found the first chaotic attractor in a simple three dimensional autonomous system [1]. For a generic three-dimensional smooth quadratic autonomous system, Sprott [2–4] found by exhaustive computer searching 19 simple chaotic systems with none, one equilibrium or two equilibria. It is very important to note that some classical 3-D autonomous chaotic systems have three particular fixed points: one saddle and two unstable saddle-foci (for example, Lorenz system [1], Chen system [5], Lü system [6], the conjugate Lorenz-type system [7]). The other 3-D chaotic systems, such as diffusionless Lorenz equations [8] and Burke–Show system [9], have two unstable saddle-foci. In 2008, Yang and Chen found another 3-D chaotic system with three fixed points: one saddle and two stable equilibria [10]. Many theoretical analyses and numerical simulations about these systems are shown in [11–21].

Without unstable equilibria, the Šilnikov condition is violated, and it is not of great significance either because that condition is known to be sufficient but certainly not necessary for chaos. In 2010, Yang et al. [22] introduced and analyzed a new 3-D chaotic system with six terms including only two quadratic terms in a form very similar to the Lorenz, Chen, Lü and Yang–Chen systems, but it has only two fixed points: two stable node-foci. Some questions about periodic, homoclinic and heteroclinic orbits and classification of chaos, are related to the dynamics of some dynamical systems. The type of chaotic systems is investigated further analytically and numerically in [23,24].

Therefore, understanding the local and the global characteristics of the chaotic dynamical systems is of great importance, because this effort often gives hints for generating/eliminating chaos and indicates the potential applications. Recently the trend of analyzing and understanding chaos has been extended to controlling and utilizing chaos. The main goal of chaos control was to eliminate chaotic behavior and to stabilize the chaotic system at one of the system's equilibrium points. More specially, when it is useful, we want to generate chaos intentionally. Until now, many advanced theories and methodologies have been developed for controlling chaos. Many scientists have more concerns with delayed control [25,26]. The existing

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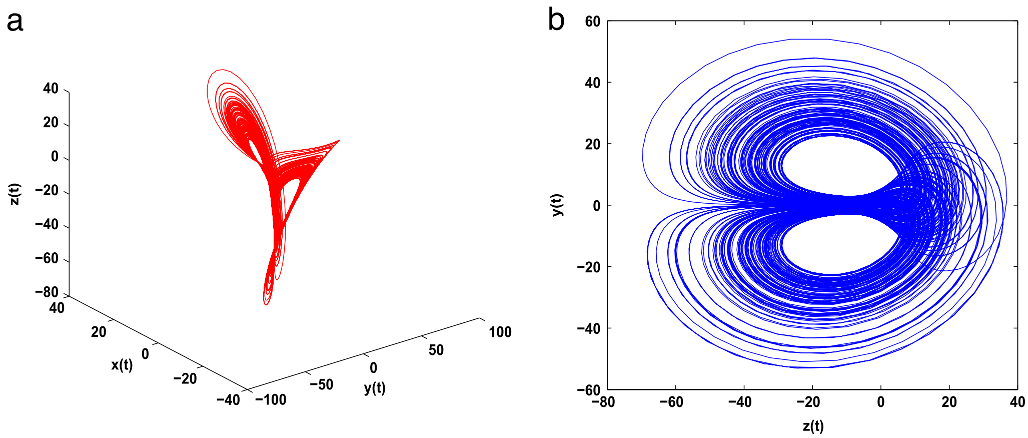


**Fig. 1.** Parameter values $(a, b, c) = (10, 100, 10.3)$ and initial values $(0.98, -1.82, -0.49)$: (a) Chaotic attractor of system (1) when the equilibria $E_{1,2}$ are both asymptotically stable; (b) Projection of (a) into $y - z$ plane.

control method can be classified, mainly, into two categories. The first one, the OGY method developed by Ott et al. [27] in the 1990s has completely changed the chaos research topic. The second one, proposed by Pyragas [28,29], used time-delayed controlling forces. Compared with the first one, it is much simpler and more convenient in controlling chaos in a continuous dynamics system. Here, we mainly study the system only with two stable node-foci proposed by Yang et al. [22] with delay. Yang et al. described this uncontrolled system by the following three-dimensional smooth autonomous system [22].

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -cy - xz \\ \dot{z} = -b + xy, \end{cases} \tag{1}$$

where $a$, $b$ and $c$ are positive real parameters. It has two equilibria

$$E_1 : (x_0, y_0, z_0) = \left(\sqrt{b}, \sqrt{b}, -c\right) \quad \text{and} \quad E_2 : (-x_0, -y_0, z_0) = \left(-\sqrt{b}, -\sqrt{b}, -c\right).$$

In particular, for parameter values $(a, b, c) = (10, 100, 10.3)$, three characteristic values of the Jacobian of the linearized equation evaluated at the equilibria point $E_{1,2}$ are: $\lambda_1 = -20.2411$, $\lambda_{2,3} = -0.0294 - 9.9402i$. The chaotic attractor and its projection in the $y - z$ plane are shown in Fig. 1(a) and (b), respectively. Therefore, system (1) has a chaotic attractor coexisting with two stable node-foci.

The purpose of the present paper is to investigate system (1) with direct time delay feedback (DTDF) analytically and numerically. Our analytical results show that the stability changes as the delays vary. Meanwhile, when all the equilibria are asymptotically stable, the chaotic attractor is converted into a stable steady state, an unstable periodic orbit or another chaotic attractor again when the delay passes through some values.

This paper is organized as follows. In Section 2, a model of system (1) with DTDF is created. The stability and the existence of Hopf bifurcation parameter are determined. In Section 3, based on the normal form method and the center manifold theorem, the direction, stability and the period of the bifurcating periodic solutions are analyzed. To verify the theoretic analysis, numerical simulations are given in Section 4. Finally, Section 5 concludes with some discussions.

## 2. System (1) with DTDF and existence of Hopf bifurcation

In this section, the controlled system by time-delayed controlling forces proposed by Pyragas [10,11] is designed as follows:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -cy - xz + k[y(t - \tau) - y] \\ \dot{z} = -b + xy, \end{cases} \tag{2}$$

where $\tau$ is the time delay, $k$ is the gain of the time delay feedback.

Due to the symmetry of $E_1$ and $E_1$, it is sufficient to analyze the stability of $E_1(x_0, y_0, z_0)$. By the linear transform

$$\begin{cases} x_1 = x - x_0, \\ y_1 = x - y_0, \\ z_1 = x - z_0, \end{cases}$$