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Existence of ground state solutions for a quasilinear Schrödinger equation with critical growth☆

Ke Wu [a,b], Fen Zhou [c,*]

[a] Department of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, PR China
[b] Department of Mathematics and Statistics, Zhaotong University, Zhaotong, Yunnan 657000, PR China
[c] Department of Information Science and Technology, Zhaotong University, Zhaotong, Yunnan 657000, PR China

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ABSTRACT

We establish the existence of ground state solutions for a quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = | u |^{22^*-2} u + \lambda | u |^{p-1} u, \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$, $1 < p < 22^* - 1$ and $\lambda > 0$ is a parameter. We do not need the subcritical exponent $p > 3$ which was used by several authors to conclude the existence of solution for the equation above.

1. Introduction and main result

This paper is concerned with the existence of ground state solutions for the critical quasilinear Schrödinger equation of the form

$$-\Delta u + V(x)u - k\Delta(u^2)u = |u|^{22^*-2}u + \lambda|u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $k = 1$, $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, $1 < p < 22^* - 1$, and $\lambda > 0$ is a parameter.

This equation can be obtained from the Schrödinger equation of the form

$$i\psi_t + \triangle\psi - W(x)\psi + k\triangle(h(|\psi|^2))h'(|\psi|^2)\psi + f(|\psi|^2)\psi = 0, \quad \text{in } [0, \infty) \times \mathbb{R}^N, \tag{1.2}$$

where $W(x)$ is a given potential, k is a real constant and f, h are real functions. Let us consider the case $h(s) = s$, $f(s) = \sqrt{|s|^{22^*-2} + \lambda|s|^{p-1}}$, $\kappa = 1$ and $W(x) = V(x) + \beta$. Putting $\psi(t, x) = \exp(-i\beta t)u(x)$, we are led immediately to Eq. (1.1).

A solution of (1.2) of the form $\psi(t, x) = \exp(-i\beta t)u(x)$ is called a standing wave. For the special case we consider, u is a solution of (1.1) if and only if ψ is a standing wave of (1.2). Hence the present research can be applied to look for a standing wave of (1.2). The quasilinear Schrödinger equation (1.2) is also derived as other models of several physical phenomena. For example, it was used for the superfluid film equation in plasma physics by Kurihara [1]. It also appears in the theory of

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* Corresponding author.
E-mail addresses: wuke2002@126.com (K. Wu), zhoufen_85@163.com (F. Zhou).

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Heidelberg ferromagnetism and magnus [2,3], in dissipative quantum mechanics [4] and in condensed matter theory [5]. Additionally, we refer readers to [6–11] and their references for further physical backgrounds of Eq. (1.1).

If $k = 0$, then (1.1) is a semilinear equation which has been studied extensively by applying directly variational methods in recent years (see e.g. [7,12,13]). If $k \neq 0$, then it is quasilinear. In this case, unlike the semilinear problem, we encounter serious additional difficulties when we try to find solutions by using the classical critical point theory. From the viewpoint of Sobolev embedding, $q = 22^* = \frac{4N}{N-2}$ is the limiting Sobolev exponent for the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$, where $X = \{u \in H^1(\mathbb{R}^N) : u^2 \in H^1(\mathbb{R}^N), \sqrt{|V|}u \in L^2(\mathbb{R}^N)\}$ is the domain of the functional corresponding to (1.1). To start with, since this embedding is not locally compact, the verification that the weak limit of a Cerami sequence is non-zero becomes much more complicated. Moreover, X being not even a vector space means the usual variational techniques cannot be directly applied.

We mention some recent mathematical studies relating to Eq. (1.1) here. In [9], by applying a change of variables, the existence of a positive solution is proved under the assumption $3 < p < 22^* - 1$. By the same change of variables as in [9], the critical quasilinear equation with periodic potential is studied in [14]. A general problem of (1.1) is the following quasilinear equation of the form

$$-\sum_{i,j=1}^{N} D_j(a_{i,j}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{N} D_s a_{i,j}(x,u)D_i u D_j u + V(x)u = |u|^{22^*-2}u + \lambda|u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $D_i = \frac{\partial}{\partial x_i}$, $D_s a_{ij}(x,s) = \frac{\partial}{\partial s} a_{ij}(x,s)$ and $V \in C(\mathbb{R}^N, \mathbb{R})$. Clearly, Eq. (1.3) is reduced to (1.1) if $a_{ij}(x,u) = (1+2u^2)\delta_{ij}$. For $\lambda = 1$ and a bounded potential V, two existence results of solutions for (1.3) were established in [15] by the Nehari method. But again the condition $3 < p < 22^* - 1$ is assumed. However, to the best of our knowledge, there is no result about the existence of solutions for (1.1) with an unbounded potential. Besides, if $p \le 3$, the existence results for (1.1) in the literature are rare (see e.g. [9,15,16]).

In this paper, we are going to study the existence of ground solutions for (1.1) with an unbounded potential. Also we are able to deal with the case when $1 < p \le 3$. By making a change of variables, the original quasilinear equation (1.1) is reduced to a semilinear one. Then the existence of ground state solutions is obtained by applying the mountain pass theorem and the concentration-compactness principle.

In order to reduce the statements of our main result, we need the following assumption on the potential V.

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf V(x) = V_0 > 0$ and for each $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \le M\} < +\infty$, where *meas* denotes the Lebesgue measure in $\mathbb{R}^N$.

A solution u of (1.1) is called a ground state solution if $u \neq 0$ and its energy is minimal among the energy of all nontrivial solutions.

Our main result is the following theorem.

Theorem 1.1. *Assume that condition* (V) *holds.*

(i) *If* $\frac{N+6}{N-2} < p < 22^* - 1$, *then* (1.1) *has a ground state solution for any* $\lambda > 0$.

(ii) *If* $1 < p \le \frac{N+6}{N-2}$, *then there exists a constant* $\lambda^* > 0$ *such that, for each* $\lambda \in (\lambda^*, \infty)$, (1.1) *has a ground state solution.*

Remark 1.1. In case (i) of our theorem, we only need $\frac{N+6}{N-2} < p < 22^* - 1$ rather than $\max\{\frac{N+6}{N-2}, 3\} < p < 22^* - 1$ which was used by several authors to conclude the existence of solutions for (1.1) (for example, see [15]). For case (ii), our result seems better than that of Theorem 1.3 in [16] for $N \ge 10$.

2. Preliminaries and minimax level

Define the Hilbert space

$$E = \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)v^2 dx < \infty \right\}$$

with the inner product

$$\langle v, w\rangle = \int_{\mathbb{R}^N} [\nabla v \cdot \nabla w + V(x)vw]dx$$

and the norm $\|v\| = \langle v, v\rangle^{1/2}$. It is well known that the embedding $E \hookrightarrow L^s(R^N)$ for $2 \le s < 2^*$ is compact under the assumption (V).

Eq. (1.1) is the Euler–Lagrange equation associated to the natural energy functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} (1+2v^2)|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2 dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |v|^{22^*} dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^N} |v|^{p+1} dx.$$