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# On the numerical solution of nonlinear option pricing equation in illiquid markets[☆]

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ABSTRACT

In this paper we focus on the numerical solution of nonlinear Black–Scholes equation modeling illiquid markets. Two monotone unconditionally stable splitting methods, ensuring positive numerical solution and avoiding unstable oscillations, are applied to solve nonlinear Black–Scholes equation modeling illiquid markets. These numerical methods are based on the LOD methods which allow us to solve the discrete equation explicitly. The properties of these methods are analyzed. The numerical results for vanilla call option are compared to the local Crank–Nicolson scheme. The numerical results for European butterfly spread are also provided.



## 1. Introduction

One of the modern financial theory's biggest successes in terms of both approach and applicability has been the Black–Scholes option pricing model developed by Fisher Black and Myron Scholes in 1973 [1] and previously by Robert Merton [2]. The celebrated Black–Scholes model is based on several restrictive assumption such as liquid, frictionless and complete markets. In recent years nonlinear Black–Scholes models have been used to build transaction costs, market liquidity or volatility uncertainty into the celebrated Black–Scholes concept. Since markets liquidity is an issue of very high concern in financial risk management, in this paper, we are interested in the option pricing model in illiquid markets proposed by Frey and Patie [3]

$$\frac{\partial V}{\partial \tau} + \frac{\sigma_0^2 S^2}{2\left(1-\rho\lambda(S)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} = 0, \quad (S,\tau) \in \Omega \times (0,T], \tag{1.1}$$

$$V(S,T) = f(S), \quad S \in \Omega := (0,+\infty), \tag{1.2}$$

where $S$ is the price of the underlying asset, $T$ is the maturity date, $\sigma_0$ is the asset volatility, $\rho$ is a parameter measuring the market liquidity, the continuous and positive function $\lambda(S)$ describes the liquidity profile of the market. The payoff function $f(S)$ is assumed to be a continuous piece-wise linear function.

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Because of the nonlinear nature of this model numerical methods are mandatory to price derivatives and portfolios. The strong nonlinearity of problem (1.1)–(1.2) makes it difficult to compute the reliable numerical solutions, so for instance in [3] perform numerical solutions not of (1.1)–(1.2) but a smoothed version of such a problem. Implicit numerical schemes have been used for numerically solving nonlinear option pricing PDEs [4], and an iterative approach is required to solve the nonlinear algebraic equation resulting from the discretization, which results in more computational cost. Consequently, Company et al. [5,6] construct explicit finite difference schemes for (1.1)–(1.2), and investigate their consistency and stability. However, these explicit schemes have the disadvantage that strictly restrictive conditions on the discretization parameters are needed to guarantee stability and positivity. To relax the restrictive conditions, in [7], Ehrhardt and Valkov propose an unconditionally stable explicit finite difference scheme based on the local Crank–Nicolson method proposed by Abduwali, Sakakihara and Niki [8] (see, also [9]). However, this method has two unfavorable factors. One is that it needs to compute the inverse of a tridiagonal matrix in every time step. This will result in low computational efficiency of this method. The other is that although it is unconditionally stable a strictly restrictive condition on the discretization parameters is needed to guarantee positivity, which is a very important issue for option pricing problems.

In this paper, we introduce an unconditionally stable splitting scheme which is of order two. This scheme does not need to compute the inverse of any matrix, and is completely explicit. For comparison, we also introduce an unconditionally stable and unconditionally positivity-preserving splitting scheme which has been successfully applied to nonlinear Black–Scholes Equation with transaction costs in [10]. But it is only of order one. The two schemes are essentially "limit" versions of the LOD methods (see Chapter IV on splitting methods in [11]) and therefore allows us to solve the discrete equation explicitly.

The manuscript is organized as follows: we begin by transforming the original equations into nonlinear diffusion equations, and considering the spatial semi-discretization and linearization of semi-discrete system. Two splitting schemes will be introduced in Section 3. The stability, the monotonicity, and the positivity-preserving property of these schemes are analyzed in Section 4. To illustrate our method we present some numerical experiments in Section 5. In this section, we compare our numerical schemes with the local Crank–Nicolson scheme given in [7] for vanilla call option. Finally, we give a summary.

## 2. Spatial semi-discretization and linearization

### 2.1. Spatial semi-discretization

For the convenience in the numerical processing and the study of the numerical analysis, we will transform the problem into a nonlinear diffusion equation. After considering the change of variable $t = T - \tau$, $U(S, t) = V(S, \tau)$, we transform the problem (1.1)–(1.2) into the following initial value problem

$$\frac{\partial U}{\partial t} - \frac{\sigma_0^2 S^2}{2\left(1 - \rho\lambda(S)S\frac{\partial^2 U}{\partial S^2}\right)^2}\frac{\partial^2 U}{\partial S^2} = 0, \quad (S, t) \in \Omega \times (0, T], \tag{2.1}$$

$$U(S, 0) = f(S), \quad S \in \Omega := (0, +\infty). \tag{2.2}$$

To numerically approximate the solution of (2.1)–(2.2) we should consider a bounded numerical domain $(S, t) \in [0, b] \times [0, T]$. According to the rigorous mathematical analysis provided by Kangro and Nicolaides in [12] in which the pointwise bounds for the error caused by various boundary conditions imposed on the artificial boundary are derived, we consider problem (2.1)–(2.2) equipped with the Dirichlet boundary conditions

$$U(0, t) = f(0), \qquad U(b, t) = f(b), \quad t \in [0, T]. \tag{2.3}$$

We introduce the spatial grid $\Omega_h$ with step $h$ by the nodes $S_i = ih,\ i = 0, \dots, M$, so that $Mh = b$. After performing the second-order central finite difference approximation of the partial derivative $\frac{\partial^2 U}{\partial S^2}(S, t)$, the so-called Gamma Greek of the option,

$$\frac{\partial^2 U}{\partial S^2} = \frac{U(S_{i+1}, t) - 2U(S_i, t) + U(S_{i-1}, t)}{h^2} + O(h^2) = \Delta_i U(t) + O(h^2) \tag{2.4}$$

we obtained the corresponding $M-1$ dimensional ODEs system for the semi-discrete solution $u(t) = [U_1(t), \dots, U_{M-1}(t)]^T$

$$u'(t) = A(u(t))u(t) + g(u(t)), \quad t \in [0, T], \tag{2.5}$$

with

$$A(u) = \frac{1}{h^2}\text{tridiag}(\beta_i(u), -2\beta_i(u), \beta_i(u)),$$

$$\beta_i(u) = \sigma_i^2(u)S_i^2, \quad \sigma_i^2(u) = \frac{\sigma_0^2}{2(1 - \rho\lambda(S_i)S_i\Delta_i u(t))^2}, \tag{2.6}$$

where $g \in \mathbb{R}^{M-1}$ is a vector, generated by the boundary conditions,

$$g(u(t)) = \frac{1}{h^2}[\beta_1 f(0), 0, \dots, 0, \beta_{M-1} f(b)]^T. \tag{2.7}$$