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Periodic solutions to a heat equation with hysteresis in the source term

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ABSTRACT

This paper is concerned with a parabolic problem with hysteresis effects in the heat source, which models the feedback control. The existence of periodic solutions is proved by the viscosity approach when the heat force changes periodically in time. More precisely, with the help of the subdifferential operator theory and the Poincaré map, the existence of solutions to the approximation problem is shown and the solution of the periodic problem is obtained under consideration by using a passage-to-limit procedure.

1. Introduction

This paper is concerned with the following system

$$\begin{cases} u_t - \triangle u + v = g \quad \text{in } Q := \Omega \times (0,\omega), \\ v_t + \partial I_u(v) \ni 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Gamma := \partial\Omega \times (0,\omega), \\ u(x,0) = u(x,\omega), \qquad v(x,0) = v(x,\omega) \quad \text{in } \Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N (1 \le N \le 3)$ with smooth boundary $\partial\Omega$, ω is a positive period, $g : Q \to \mathbb{R}$ is given ω-Periodic function, and $\partial I_u(v)$ is the subdifferential of the indicator function $I_u(v)$ specified by

$$\partial I_u(v) = \begin{cases} \emptyset & \text{if } v > f^*(u) \text{ or } v < f_*(u), \\ [0,+\infty) & \text{if } v = f^*(u) > f_*(u), \\ \{0\} & \text{if } f_*(u) < v < f^*(u), \\ (-\infty, 0] & \text{if } v = f_*(u) < f^*(u), \\ \mathbb{R} & \text{if } v = f^*(u) = f_*(u), \end{cases} \tag{1.2}$$

where $f_*, f^*\colon \mathbb{R} \to \mathbb{R}$ are prescribed non-decreasing and bounded functions with $f_* \le f^*$.

In the case that u represents the temperature, problem (1.1) can be regarded as the model for heat conduction with feedback control, which has the typical hysteresis memory property (see [1]). In fact, Eq. $(1.1)_2$ (the second equation of problem (1.1)) is equivalent to the input–output relation of a hysteresis operator generated by two curves $v = f_*(u)$ and $v = f^*(u)$ as illustrated in Fig. 1 (see Visintin [2] and Brokate and Sprekels [3]).

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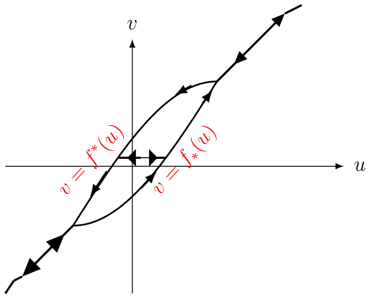

Fig. 1. Input–output relation.

Hysteresis operator naturally arises in mathematical description of many physical processes, such as phase transitions, superconductivity and biology [4,3,5–7,2] and becomes an important concept to analyse the mathematical properties of some hysteresis models [3,8–10,2]. According to [2], the hysteresis is referred to as a rate independent memory effect. More precisely, let us consider a system which is described by the input u and output v. The memory effect means that at any instant t the value of the output is not simply determined by the value $u(t)$ of the input at the same instant, but it depends also on the previous evolution of the input u. The rate independence means that the path of the couple $(u(t); v(t))$ is invariant with respect to any increasing time homeomorphism. As pointed out by Visintin in [2], some kinds of hysteresis operators can be represented by ordinary differential inclusion containing subdifferential of the indicator function of a closed interval. In particular, the function v in (1.1) is determined by a hysteresis operator, which can be referred to [11,12] for a brief explanation of this fact. Models involving ordinary differential equations with hysteresis were considered by many authors and are nowadays relatively well investigated, see e.g., [13–15,6,16–18].

Partial differential equations with hysteresis have been extensively studied during the last decades ([11,19,3,20–22,2,23] and references therein), but nevertheless there seems to be a few results related with the periodic behaviour of solutions. In fact, A. Friedman and L. Jiang [20] studied the periodic solutions of the heat equation

$$u_t(x,t) = u_{xx}(x,t), \quad (x,t) \in (0,1) \times (0,+\infty)$$

with hysteresis operator of relay type in Robin boundary conditions. Also in [24], I. Götz et al. showed the existence of periodic solutions to the one-dimensional Stefan problem with Dirichlet boundary conditions involving relay hysteresis. It is noted that by reducing the original problem to an infinite-dimensional dynamical system, P. Gurevich, S. Tikhomirov [25,26] recently considered the periodic problem to the multi-dimensional heat equation with hysteresis on the boundary

$$\begin{cases} u_t(x,t) = \triangle u(x,t) \quad (x,t) \in Q, \\ \dfrac{\partial u}{\partial \nu} = K(x)v(t) \quad x \in \Gamma,\ t \in (0,\infty), \end{cases} \tag{1.3}$$

where $K \in C^\infty(\partial\Omega)$ is a real-valued function, and $v(t)$ satisfies the ordinary differential equation

$$av'(t) + v(t) = \mathcal{H}(\hat{u})(t)$$

with $a \geq 0$ and a relay operator $\mathcal{H}$, $\hat{u} = \int_\Omega m(x)u(x,t)dx$, $m \in L^2(\Omega)$ is a real-valued weight function determined by characteristics of the thermal sensors. For the heat equation with hysteresis in the source of the form

$$u_t - \triangle u + \mathcal{F}(u) = f,$$

where the hysteresis operator $\mathcal{F}(u)$ is global bounded and Lipschitz continuous, J. Kopfova [21] applied Leray–Schauder fixed point theorem to prove the existence of periodic solutions.

It is worth to mention that system (1.1) can be regarded as the periodic problem to a sort of quasi-variational evolution inequalities. In fact, the equations in (1.1) consist a special class of quasi-variational inequality, since the constraint $f_*(u(t)) \leq v(t) \leq f^*(u(t))$ in the second equation depends on an unknown function u, which, in turn, is determined by the function v via the first equation and boundary condition in (1.1). We refer to [27–32] and the references therein for the papers dealing with the quasi-variational problem. One of main difficulties in the mathematical treatment comes from the quasi-variational structure. Indeed, the characterization of quasi-variational inequalities renders the argument in [25,26] invalid.

Motivated by the above works, the aim of present paper is to show the existence of periodic solutions to (1.1) by the viscosity approach. More precisely, since $\partial I_u(v)$ is the non-smooth function which depends on u, we consider the following approximate problem for $0 < \varepsilon < 1$

$$\begin{cases} u_t - \triangle u + v = g \quad \text{in } Q, \\ v_t - \varepsilon\triangle v + \partial I_u^\varepsilon(v) = 0 \quad \text{in } Q, \\ u = 0, \qquad v = 0 \quad \text{on } \Gamma, \\ u(x,0) = u(x,\omega), \qquad v(x,0) = v(x,\omega) \quad \text{in } \Omega, \end{cases} \tag{1.4}$$