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Existence of solutions for generalized equilibrium problem in G-convex space

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ABSTRACT

In this paper, we introduce and study a kind of generalized equilibrium problem in a G-convex space. By means of the fixed-point theorems, we obtain some existence theorems of solutions for the generalized equilibrium problems.

1. Introduction

Let X, Y, D be three nonempty sets and Z a topological space, $S : X \multimap Y$, $T : X \multimap D$ and $F : X \times Y \times D \multimap Z$ set-valued mappings. Let C be a set-valued mapping defined as $C : X \multimap Z$ with $\text{int}C(x) \neq \emptyset$, where $\text{int}C(x)$ denotes the topological interior of $C(x)$. Let $\xi : Y \to X$ be a single-valued mapping. A generalized equilibrium problem (GEP) discussed in this paper is to find $(\bar{x} \times \bar{t}) \in X \times D$ such that

$$\bar{x} \in \xi(S(\bar{x})), \bar{t} \in T(\bar{x}) \quad \text{and} \quad F(\bar{x}, y, \bar{t}) \not\subseteq -\text{int}C(\bar{x}), \quad \forall y \in S(\bar{x}). \tag{1}$$

This generalized equilibrium problem includes some models of generalized equilibrium problems studied by many researches, see Refs. [1–13] and references therein. The following problems are the special cases of GEP (1).

(A) If X is a real topological vector space, $Y = X$ and $\xi : X \to X$ is the identity mapping, that is, $\xi(x) = x$ for each $x \in X$, then GEP (1) reduces to the generalized vector equilibrium problem ($GVEP$) of finding $\bar{x} \in S(\bar{x})$ such that

$$\exists \bar{t} \in T(\bar{x}), F(\bar{x}, y, \bar{t}) \not\subseteq -\text{int}C(\bar{x}), \quad \forall y \in S(\bar{x}), \tag{2}$$

which is similar to the generalized vector quasiequilibrium problem ($GVQEP$) studied by Lin et al. in Ref. [11]. GVQEP includes the quasiequilibrium problem (QEP) considered by Hai and Khanh in Ref. [8] as it is a special case. The generalized vector quasi-variational-like inequalities studied by Xiao and Liu in Ref. [14] is a special case of GVEP (2).

(B) In addition, if $S(x) \equiv X$ for all $x \in X$, then GVEP (2) reduces to the problem of finding $\bar{x} \in X$ such that

$$\exists \bar{t} \in T(\bar{x}), F(\bar{x}, y, \bar{t}) \not\subseteq -\text{int}C(\bar{x}), \quad \forall y \in X, \tag{3}$$

which is the generalized vector equilibrium problem studied by Lin in Ref. [5]. GVEP (3) includes the vector equilibrium problem (VEP) studied by many researchers, for example, in Refs. [15–17].

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(C) If D is a singleton and $S(x) \equiv Y$ for all $x \in X$, then GEP (1) reduces to the problem of finding $\bar{x} \in X$ such that

$$F(\bar{x}, y) \nsubseteq -\mathrm{int}C(\bar{x}), \quad \forall y \in Y, \tag{4}$$

which is the abstract generalized vector equilibrium problem (*AGVEP*) studied by Ding and Park in Ref. [2]. The scalar equilibrium problem in generalized convex spaces considered by Mitrovic in Ref. [18] is a special case of AGVEP in Ref. [2].

According to the above arguments, for a suitable choice of the spaces X, Y, Z, D and the mappings S, T in GEP (1), we can obtain a number of known classes of generalized vector equilibrium problems, generalized vector quasiequilibrium problems, vector equilibrium problems and vector variational inequalities etc.

In this paper, we establish some existence results for the generalized equilibrium problems in a G-convex space using some fixed-point theorems.

2. Preliminaries

Let X, Y be two topological spaces. A set-valued mapping $T : X \multimap Y$ is said to *have open lower sections* if its fibers $T^{-}(y) = \{x \in X : y \in T(x)\}$ are open in X for every $y \in Y$. If T is single-valued, then T^{-} is usually denoted by T^{-1}.

A *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set $D \subseteq X$ and a set-valued mapping $\Gamma : \mathcal{F}(D) \multimap X$ such that, for each $A = \{x_0, \dots, x_n\} \in \mathcal{F}(D)$ with the cardinality $|A| = n + 1$, there exists a continuous mapping $\Phi_A : \Delta_n \to \Gamma(A)$ such that $J \subseteq A$ implies $\Phi_A(\Delta_J) \subseteq \Gamma(J)$, where, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^{n}$ and Δ_J the face of Δ_n corresponding to $J \in \mathcal{F}(A)$; that is, if $J = \{x_{i_0}, x_{i_1}, \dots, x_{i_k}\}$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$ is the convex hull of the vertices $\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. $\mathcal{F}(D)$ denotes the family of all nonempty finite subsets of D. For the sake of convenience, we write it as $(X; \Gamma) = (X, X; \Gamma)$.

A subset K of the G-convex space $(X, D; \Gamma)$ is said to be *G-convex* if $\Gamma(A) \subseteq K$ for each $A \in \mathcal{F}(K \cap D)$. Define the *G-convex hull* of K as $G\text{-co}(K) = \bigcap\{B : K \subseteq B, B \text{ is } G\text{-convex}\}$. For details on the G-convex spaces, see Refs. [2,19,20] and the references therein.

Definition 2.1. Let X be a set and $(Y; \Gamma)$ a G-convex space. A mapping $F : X \multimap Y$ is called *G-convex* if $F(x)$ is a G-convex subset of Y for every $x \in X$.

If Y is a topological vector space, define $\Gamma(A) = \mathrm{co}(A)$ for $A \in \mathcal{F}(Y)$, then $(Y; \Gamma)$ is a G-convex space. In this case, if the mapping F is G-convex, then it is a *convex set-valued mapping*, that is, $F(x)$ is a convex subset of Y for each $x \in X$.

Lemma 2.1. *If* (X, Γ) *is a G-convex space and* $K \subseteq X$*, then*

(i) $\Gamma(M) \subseteq G\text{-co}(K)$ *for any* $M \in \mathcal{F}(K)$.
(ii) $G\text{-co}(M) \subseteq G\text{-co}(K)$ *for any* $M \in \mathcal{F}(K)$.
(iii) *for each* $x \in G\text{-co}(K)$*, there is a finite set* $\{x_1, x_2, \dots, x_n\} \subseteq K$ *such that* $x \in G\text{-co}\{x_1, x_2, \dots, x_n\}$.

Proof. Noting that G-co(K) is G-convex and $K \subseteq G\text{-co}(K)$, we have that $\Gamma(M) \subseteq G\text{-co}(K)$ for any $M \in \mathcal{F}(K) \subseteq \mathcal{F}(G\text{-co}(K))$, that is, (i) holds. From the definition of the G-convex subset, we deduce directly that (ii) holds.

Let $B = \bigcup_{N \in \mathcal{F}(K)} G\text{-co}(N)$, then $K \subseteq B$. Here we show that B is G-convex. Assume that $N \in \mathcal{F}(B)$, then there are $N_1, N_2, \dots, N_k \in \mathcal{F}(K)$ such that $N \subseteq \bigcup_{i=1}^{k} G\text{-co}(N_i)$. Since $\bigcup_{i=1}^{k} N_i \in \mathcal{F}(K)$, then $G\text{-co}\left(\bigcup_{i=1}^{k} N_i\right) \subseteq B$ and $G\text{-co}(N_j) \subseteq G\text{-co}\left(\bigcup_{i=1}^{k} N_i\right)$ for each $j \in \{1, 2, \dots, k\}$ by (ii). Thus $\bigcup_{i=1}^{k} G\text{-co}(N_i) \subseteq G\text{-co}\left(\bigcup_{i=1}^{k} N_i\right)$. Since $G\text{-co}\left(\bigcup_{i=1}^{k} N_i\right)$ is G-convex, then $\Gamma(N) \subseteq G\text{-co}\left(\bigcup_{i=1}^{k} N_i\right) \subseteq B$. Hence, B is G-convex and $G\text{-co}(K) \subseteq B$. Thus, taking $x \in G\text{-co}(K)$, there exists $\{x_1, x_2, \dots, x_n\} \subseteq K$ such that $x \in G\text{-co}\{x_1, x_2, \dots, x_n\}$. □

Definition 2.2. Let $(X; \Gamma)$ be a G-convex space, Y and Z two nonempty sets. Let $F : X \times Y \multimap Z$ and $C : X \multimap Z$ be set-valued mappings.

(i) $F(x, y)$ is said to be *generalized G-diagonally quasiconvex* in y with respect to C if, for each $B = \{y_1, y_2, \dots, y_n\} \in \mathcal{F}(Y)$, there exists $A = \{x_1, x_2, \dots, x_n\} \subseteq \mathcal{F}(X)$ such that, for any $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subseteq A$ and $x \in G\text{-co}\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$, there exists $j \in \{1, \dots, k\}$ such that $F(x, y_{i_j}) \nsubseteq C(x)$.
(ii) If $Y = X$, $F(x, y)$ is said to be *G-diagonally quasiconvex* in y with respect to C if, for any $A = \{x_1, x_2, \dots, x_n\} \in \mathcal{F}(X)$ and $x \in G\text{-co}(A)$, there exists $j \in \{1, \dots, n\}$ such that $F(x, x_j) \nsubseteq C(x)$.
(iii) Let $\xi : Y \to X$ be a single-valued mapping. $F(x, y)$ is said to be *ξ-G-diagonally quasiconvex* in y with respect to C if, for any $\{y_1, y_2, \dots, y_n\} \in \mathcal{F}(Y)$ and $x \in G\text{-co}\{\xi(y_1), \xi(y_2), \dots, \xi(y_n)\}$, there exists $j \in \{1, \dots, n\}$ such that $F(x, y_j) \nsubseteq C(x)$.

Remark 2.3. From Definition 2.2, it is easy to verify the following two propositions.

(I) If $\xi : Y \to X$ is injective, then ξ-G-diagonally quasiconvex implies generalized G-diagonally quasiconvex.
(II) If $\xi : X \to X$ is the identity mapping and $Y = X$, then ξ-G-diagonally quasiconvex is equivalent to G-diagonally quasiconvex.