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Some preserving subordination and superordination of analytic functions involving the Liu–Owa integral operator

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ABSTRACT

In this paper, we obtain some subordination and superordination-preserving results of analytic functions involving the Liu–Owa integral operator. Sandwich-type result is also obtained.

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Let $A(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\};\ z \in U). \tag{1.1}$$

Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [1,2]).

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{1.3}$$

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then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [1,2]).

Motivated essentially by Jung et al. [3], Liu and Owa [4] introduced the integral operator $Q^{\alpha}_{\beta,p} : A(p) \to A(p)$ as follows:

$$Q^{\alpha}_{\beta,p}f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt, \quad (\alpha > 0;\ \beta > -1;\ p \in \mathbb{N}), \tag{1.4}$$

and

$$Q^{0}_{\beta,p}f(z) = f(z), \quad (\alpha = 0;\ \beta > -1).$$

For $f \in A(p)$ given by (1.1), then from (1.4), we deduce that

$$Q^{\alpha}_{\beta,p}f(z) = z^p + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{n=1}^{\infty} \frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)} a_{p+n} z^{p+n} \quad (\alpha \ge 0;\ \beta > -1; p \in \mathbb{N}). \tag{1.5}$$

It is easily verified from the definition (1.5) that (see [4])

$$z\left(Q^{\alpha}_{\beta,p}f(z)\right)' = (\alpha+\beta+p-1)\, Q^{\alpha-1}_{\beta,p}f(z) - (\alpha+\beta-1)\, Q^{\alpha}_{\beta,p}f(z). \tag{1.6}$$

We note that $Q^{1}_{c,p}f(z) = J_{c,p}(f)(z) = \frac{c+p}{z^c}\int t^{c-1}f(z)dt\ (c > -p)$, where the operator $J_{c,p}$ is the generalized Bernardi–Libera–Livingston integral operator (see [5]). Also, we note that the one-parameter family of integral operator $Q^{\alpha}_{\beta,1} = Q^{\alpha}_{\beta}$ was defined by Jung et al. [3] and studied by Aouf [6] and Gao et al. [7].

To prove our results, we need the following definitions and lemmas.

Definition 1 ([1]). Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U} \setminus E(q)$ where

$$E(q) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0) \equiv \mathcal{F}_0$ and $\mathcal{F}(1) \equiv \mathcal{F}$.

Definition 2 ([2]). A function $L(z,t)$ $(z \in U, t \ge 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in U for all $t \ge 0$, $L(z,0)$ is continuously differentiable on [0; 1) for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \le t_1 \le t_2$.

Lemma 1 ([8]). *The function $L(z,t) : U \times [0;1) \longrightarrow \mathbb{C}$ of the form*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \quad (a_1(t) \neq 0;\ t \ge 0),$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in U,\ t \ge 0).$$

Lemma 2 ([9]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re}\{H(is;t)\} \le 0$$

for all real s and for all $t \le -n\left(1+s^2\right)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in U and

$$\operatorname{Re}\left\{H\left(p(z); zp'(z)\right)\right\} > 0 \quad (z \in U),$$

then $\operatorname{Re}\{p(z)\} > 0$ for $z \in U$.

Lemma 3 ([10]). *Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ $(z \in U)$, then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in U and satisfies $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ for $z \in U$.