Computers and Mathematics with Applications 62 (2011) 3642–3647

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A necessary condition of viability for fractional differential equations with initialization

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ARTICLE INFO

Article history:
Received 2 April 2011
Received in revised form 7 September 2011
Accepted 8 September 2011

Keywords:
Viability
Fractional derivative
Fractional differential equation

ABSTRACT

In this paper, viability results for nonlinear fractional differential equations with the Riemann–Liouville derivative are proved. We give a necessary condition for fractional viability of a locally closed set with respect to a nonlinear function.

1. Introduction

One of the most important problems in the theory of differential equations, classical or fractional ones, is that of local existence. This consists in checking whether or not, for each initial datum, the Cauchy problem attached to a given differential equation has at least one solution. The first meaningful local existence result is due to Cauchy and refers to the case in which $K = D, f : I \times K \to \mathbb{R}^n$ is a C^1-function, D is the domain of f i.e. a nonempty and open subset $D \subseteq \mathbb{R}^n$, I is an open interval in $\mathbb{R}$ and K is a nonempty subset in $\mathbb{R}^n$. This was extended by Lipschitz to the class of all functions f satisfying the homonymous condition, and by Peano [1] to general continuous functions. But there are also situations in which instead of a domain D one has to consider a set $K \neq D$ and the state x of a certain system must evolve within a given closed subset K in $\mathbb{R}^n$. These considerations lead to the concept of viability of a set K with respect to a given function f. Our motivation for considering these kind of problems for fractional differential equations was the value of viability theory. As the author in [2], the most important book on classical viability theory, says: *viability theory is a mathematical theory that offers mathematical metaphors of evolution of macrosystems arising in biology, economics, cognitive sciences, games, and similar areas, as well as in nonlinear systems of control theory.*

In 1942, Nagumo [3] formulated a necessary and sufficient condition for viability, namely, condition under which all trajectories of a vector field starting at points of a closed set K (constraint set) stay in this set. In the original result of Nagumo and in its later generalizations (see for example [4–8]), the tangent cone to the set K was used to express the conditions of viability. In this paper, we also use the concept of contingent cone to formulate a necessary condition of viability for fractional equations with the Riemann–Liouville derivative. In our previous work [9], we gave a sufficient condition for a fractional system to be viable with respect to a closed set K but we modified the definition of the contingent cone and got the cone that met our demands.

Our main idea uses the concept of initialization process, that can be found more exactly in papers [10–12]. Thanks to some modifications of the tangency point, we are able to use the classical definition of the contingent cone as a tool in necessary condition of viability in the present work.

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.09.017

We do not prove a sufficient condition of viability in the paper. In our opinion, to get a sufficient condition one needs to follow the construction of the ε-solutions similar to the one presented by the authors in [9]. However, in the case of a modified initial value problem with the Riemann–Liouville derivative, that is considered here, it demands additional research to be done in a separate paper.

2. Preliminaries

In this section, we make a review of notations, definitions, and some preliminary facts which are useful for the paper. We recall definitions of fractional integrals of arbitrary order, the Riemann–Liouville derivative of order $q \in (0, 1)$, and a description of special functions in the fractional calculus.

Definition 1 (*[13–15]*)**.** Let $\varphi \in L_1([0, t_1], \mathbb{R})$. The integral

$$(I_{0+}^{q}\varphi)(t) = \frac{1}{\Gamma(q)}\int_0^t \varphi(s)(t-s)^{q-1}ds, \quad t > 0,$$

where Γ is the gamma function and $q > 0$, is called, the *left-sided fractional integral of order q*. Additionally, we define $I_{0+}^{0} := \mathbf{I}$ (identity operator).

Remark 2 (*[16]*)**.** We write $I_{0+}^{q} = I^q$ and then $(I^q f)(t) = (f * \varphi_q)(t)$, where $\varphi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$ for $t > 0$, $\varphi_q(t) = 0$ for $t \leq 0$, and $\varphi_q \to \delta(t)$ as $q \to 0$, with δ the delta Dirac pseudo function.

Moreover, fractional integration has the following property

$$I_{0+}^{q}(I_{0+}^{p}\varphi) = I_{0+}^{q+p}\varphi, \quad q \geq 0,\ p \geq 0. \tag{1}$$

The best well known fractional derivatives are the Riemann–Liouville and the Caputo ones.

Definition 3 (*[13,14]*)**.** Let φ be defined on the interval $[0, t_1]$. The *left-sided Riemann–Liouville derivative of order q and the lower limit* 0 is defined through the following:

$$(D_{0+}^{q}\varphi)(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t \varphi(s)(t-s)^{n-q-1}ds,$$

where n is a natural number satisfying $n = [q] + 1$ with $[q]$ denoting the integer part of q.

Remark 4. If $q \in (0, 1)$, then the left-sided Riemann–Liouville fractional derivative of order q becomes

$$(D_{0+}^{q}\varphi)(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t \varphi(s)(t-s)^{-q}ds = \frac{d}{dt}\left(\left(I_{0+}^{1-q}\varphi\right)(t)\right).$$

From [13, Theorem 2.4], we have the following properties.

Proposition 5. *If $q > 0$, then $(D_{0+}^{q}(I_{0+}^{q}\varphi))(t) = \varphi(t)$ for any $\varphi \in L_1(0, t_1)$, while*

$$\left(I_{0+}^{q}(D_{0+}^{q}\varphi)\right)(t) = \varphi(t)$$

is satisfied for $\varphi \in I_{0+}^{q}(L_1(0, t_1))$ with

$$I_{0+}^{q}(L_1(0, t_1)) = \{\varphi(t) : \ \varphi(t) = \left(I_{0+}^{q}\psi\right)(t),\ \psi \in L_1(0, t_1)\}.$$

However,

$$\left(I_{0+}^{q}(D_{0+}^{q}\varphi)\right)(t) = \varphi(t) - \sum_{k=0}^{n-1}\frac{t^{q-k-1}}{\Gamma(q-k)}\left(I_{0+}^{n-q}\varphi\right)(t)\,|_{t=0}\,.$$

In particular, for $q \in (0, 1]$ we have

$$\left(I_{0+}^{q}(D_{0+}^{q}\varphi)\right)(t) = \varphi(t) - \frac{t^{q-1}}{\Gamma(q)}\left(I_{0+}^{1-q}\varphi\right)(t)\,|_{t=0}\,.$$

The following formulas are useful:

$$I_{0+}^{q}t^p = \frac{\Gamma(p+1)}{\Gamma(p+q+1)}t^{p+q} \textit{and} \qquad D_{0+}^{q}t^p = \frac{\Gamma(p+1)}{\Gamma(p-q+1)}t^{p-q}.$$