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Optimal control of partial differential equations based on the Variational Iteration Method

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A B S T R A C T

In this work, the Variational Iteration Method is used to solve a quadratic optimal control problem of a system governed by linear partial differential equations. The idea consists in deriving the necessary optimality conditions by applying the minimum principle of Pontryagin, which leads to the well-known Hamilton–Pontryagin equations. These linear partial differential equations constitute a multi-point-boundary value problem. To achieve the solution of the Hamilton–Pontryagin equations using the Variational Iteration Method, an approach is proposed and illustrated by two application examples.

1. Introduction

Unlike the optimal control theory of systems governed by partial differential equations (PDEs) [1–3], optimal control of systems governed by ordinary differential equations (ODEs) is relatively advanced [4,5]. For this kind of system, efficient approaches have been developed (calculus of variations, minimum principle of Pontryagin and dynamic programming), and the solution is achieved in general numerically since the optimality conditions are a set of nonlinear ODEs (Euler–Lagrange equation, Hamilton–Pontryagin equations or Hamilton–Jacobi Bellman equation) [4]. For systems described by PDEs, even though these approaches can be applied, but obtaining the solution, either numerically or analytically, is a difficult task due to the complexity of the calculations to handle [1,3,6–9].

Recently, various methods that provide an approximate analytical of both linear and nonlinear differential equations have been developed in the literature [10]. These methods use practical iterative formulas to provide the solution which may converge to the exact solution if it exists otherwise an approximate analytical solution can be obtained by performing only few iterations. The well-known and established methods are Adomian's Decomposition Method [11], the Homotopy Perturbation Method [12] and the Variational Iteration Method [13].

These methods have been successfully applied to solve optimal control problem of systems governed by ODEs [14–19]. To the best of our knowledge, apart from the contribution of [20], that used the Variational Iteration Method (VIM), these methods are still not applied in the area of optimal control of systems described by PDEs, which motivates this work.

The VIM method developed by [21] is a powerful mathematical tool that provides iteratively the solution of a wide class of both ODEs and PDEs [22–26]. The method provides an approximate analytical solution of differential equations in the

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form of an infinite series [13]. The terms of the series are determined using correction functional that involves the Lagrange multiplier [27], as a key element, identified using the calculus of variations theory. In addition, the VIM method starts by considering an initial solution, which is chosen so that the boundary conditions are verified to ensure a rapid convergence [28]. Note that, the convergence of the VIM method has been investigated by many authors [28–31] and the main existing results are developed based on the Banach fixed point theorem [32].

In this paper the VIM method is used to solve the quadratic optimal control problem of systems governed by linear PDEs. The method is applied to achieve the solution of the Hamilton–Pontryagin equations, which constitute the necessary optimality conditions derived using the minimum principle of Pontryagin. Our choice of the VIM method is motivated by the fact that this method needs less effort compared to the Adomian Decomposition Method (ADM) and the Homotopy Perturbation Method (HPM) that need the calculation of the Adomian polynomials, which constitutes a tedious task [33]. In addition, using the VIM method, the solution is achieved using a practical correction functional that can provide either an exact or an approximate analytical solution according to the complexity of the equation. The main advantage of the VIM method over the alternative methods (ADM and HPM) is the existence of sound results concerning its convergence [28,29].

The paper is organized as follows. The linear quadratic optimal control problem is formulated in Section 2. Section 3 summarizes the necessary optimality conditions of the formulated optimal control problem derived using the Pontryagin's minimum principle. In Section 4, the solution of the PDEs using the VIM method is presented. Section 5 is devoted to the application of the VIM method for solving the Hamilton–Pontryagin equations. Application examples are given in Section 6 and the last section is reserved to the conclusion.

2. Linear quadratic optimal control problem

The linear quadratic optimal control problem considered in this work is formulated as follows:

$$\min_{u(z,t)} J(u(t)) = \int_0^{t_f} \int_0^{l} \left[q \left(x^d(z,\ t) - x(z,\ t) \right)^2 + r\, u^2(z,\ t) \right] dz\, dt \tag{1}$$

Subject to

$$x_t(z,\ t) = f\left(x(z,\ t),\ x_z(z,\ t),\ x_{zz}(z,\ t),\ u(z,\ t),\ z,\ t\right), \tag{2}$$

with the initial condition

$$x(z, 0) = x_0(z), \tag{3}$$

and the boundary conditions

$$\alpha_0\, x(0,\ t) + \beta_0\, x_z(0,\ t) = h_0(t), \tag{4}$$

$$\alpha_l\, x(l,\ t) + \beta_l\, x_z(l,\ t) = h_l(t). \tag{5}$$

The final state $x(z,\ t_f)$ can be fixed (specified), that is,

$$x(z,\ t_f) = x_f(z) \tag{6}$$

or free (unspecified) according to the formulation of the optimal control problem.

In the formulated optimal control problem, $x(z,\ t)$ is the state variable assumed to be sufficiently smooth of its arguments z and t, $u(z, t)$ is the control variable, $x_t(z,\ t), x_z(z,\ t)$ are the partial derivatives of $x(z,\ t)$ with respect to t and z, respectively. $x_{zz}(z,\ t)$ is the second partial derivative of $x(z,\ t)$ with respect to z. $z \in [0,\ l]$ and $t \in [0,\ t_f]$ are the spatial and the temporal independent variables, respectively. t_f is the final time assumed to be fixed. $x_0(z)$ is the initial state, $x^d(z,\ t)$ is the desired profile and f is a linear function. q and r are positive weighting factors. α_0, α_l, β_0 and β_l are constants. $h_0(t)$ and $h_l(t)$ are smooth functions.

In the following, it is assumed that $u(z,\ t)$ belongs to the space of admissible controls denoted by $\mathcal{U}$ and the state $x(z,\ t)$ belongs to the space of the reachable state $\mathcal{X}$. In addition, assume that the quadratic function J, defined on $\mathcal{U}$, is continuous and strictly convex, it is also assumed that the set $\mathcal{U}$ is compact, which means that the formulated optimal problem is well-posed and admits a unique solution [34].

The objective is to determine the control variable $u(z,\ t)$ that minimizes the performance index (1) and satisfies the constraints (2)–(6) if the final state is fixed.

3. Necessary optimality conditions

The minimum principle of Pontryagin is an elegant method that allows deriving the necessary optimality conditions for the optimal control problems, that is, the well-known Hamilton–Pontryagin equations and their solution yields the optimal control law. According to the minimum principle of Pontryagin, the solution of the optimal control problem is determined by minimizing the Hamiltonian defined, by introducing the costate variable $p(z,\ t)$, as follows:

$$H(x,\ x_z,\ x_{zz}, u,\ p,\ z,\ t) = q\left[x^d(z,\ t) - x(z,\ t)\right]^2 + r\, u^2(z,\ t) + p(z,\ t) f(x,\ x_z,\ x_{zz}, u,\ z,\ t), \tag{7}$$