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Existence and multiplicity of weak solutions for a singular quasilinear elliptic equation

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ABSTRACT

In this paper, we consider the singular quasilinear elliptic equation

$$-\operatorname{div}\left(|x|^{-ap}|Du|^{p-2}Du\right)+V(x)|u|^{p-2}u = h(x)|u|^{q-2}u+\lambda H(x)|u|^{r-2}u,\quad x\in\mathbb{R}^N\setminus\{0\},\tag{0.1}$$

where $\lambda>0, -\infty<a<\frac{N-p}{p}, 1<p<N, 1<q,r<p^*=\frac{Np}{N-p}$. The function $V(x)$ is continuous and nonnegative; the functions $h(x), H(x)\in L^\infty(B_{R+1}\setminus B_\tau)$ with $0<\tau<R_0<R<+\infty$. Using variational methods, we obtain a number of results on existence and multiplicity of solutions by considering all possible orderings of exponents p, q, r.

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1. Introduction

Our aim in the present work is to consider weak solutions of the quasilinear elliptic equation

$$-\operatorname{div}\left(|x|^{-ap}|Du|^{p-2}Du\right)+V(x)|u|^{p-2}u = h(x)|u|^{q-2}u+\lambda H(x)|u|^{r-2}u,\quad x\in\mathbb{R}^N\setminus\{0\}\tag{1.1}$$

where $\lambda>0, -\infty<a<\frac{N-p}{p}, 1<p<N, 1<q,r<p^*=\frac{Np}{N-p}$. The function $V(x)$ is continuous, nonnegative and its vanishing set $V_0:=\{x\in\mathbb{R}^N : V(x)=0\}\subset B_{R_0}$ for some finite $R_0>0$. The functions $h(x), H(x)\in L^\infty(B_{R+1}\setminus B_\tau)$ with $0<\tau<R_0<R<+\infty$.

The existence and multiplicity of solutions for a singular elliptic equation with weights like (1.1) in a bounded domain Ω with zero Dirichlet data have been widely studied by several authors; see [1–6] and references therein.

Xuan in [1] established a compact embedding theorem which is an extension of the classical Rellich–Kondrachov compact embedding theorem and then consider the solvability of Eq. (1.1) with $V(x)=-\lambda|x|^{-(a+1)p+c}$, $h(x)=|x|^{-bq}$ and $H(x)=0$. In [2], some multiplicity results are presented for the problem

$$-\operatorname{div}(|x|^{-2a}Du)=\lambda|x|^{-2b}f(u)+\mu|x|^{-2c}g(u)\quad\text{in }\Omega$$

with null boundary conditions. Rodrigues [4], Ji [5] and Chen et al. [6] have established the existence and multiplicity of nonnegative solutions for singular elliptic equations like Eq. (1.1) respectively via Nehari manifold and fibering maps.

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For the solvability of quasilinear elliptic equations with singular weights like Eq. (1.1) in $\mathbb{R}^N$, we refer the reader to [7–11]. Assunçâo et al. in [7] studied the multiplicity of solutions for singular equation (1.1) with $V(x) = 0$, $h(x) = \alpha|x|^{-bq}$, $H(x) = \beta|x|^{-dr}k(x)$. Recently, Su and Wang in [8] studied the existence of entire solutions of nonlinear elliptic equations of the form

$$-\mathrm{div}(A(|x|)|Du|^{p-2}Du) + V(|x|)|u|^{p-2}u = Q(|x|)f(u) \quad \text{in } \mathbb{R}^N.$$

In [8], on one hand, they extended some of the recent embedding results for radially symmetric functions to more general cases allowing in particular weighted p-Laplacian operators involved here; on the other hand, their results further revealed relations between the growth and decay of the potentials and the ranges of embeddings. Similar considerations can be found in [12,13].

The starting point of the variational approach to problems like (1.1) is the weighted Sobolev–Hardy inequality (see (2.3)). Hence the potentials are singular at the origin in the references mentioned above. Moreover, the potentials are radial in most cases.

Motivated by [1,8,14], we shall consider the existence and multiplicity of Eq. (1.1) with the potentials which are not necessarily radial, can be unbounded or decaying to zero as $|x| \to +\infty$ and can be singular or bounded at the origin.

The rest of the paper is organized as follows. In Section 2, we present the variational framework and establish pertinent embedding theorems involving weighted spaces that are used repeatedly in the sequel. In Section 3, we give some properties of the Nehari manifold and fibering maps. Section 4 is devoted to multiplicity of weak solutions for Eq. (1.1) where the nonlinearities are concave and convex, i.e., the exponents satisfy $1 < q < p < r < p^*$ or $1 < r < p < q < p^*$. In Section 5, we prove existence theorems by considering other possible orderings of the exponents p, q, r.

2. Embedding theorems

We first introduce some spaces. For any $s \in (1, +\infty)$ and any continuous function $K(x) : \mathbb{R}^N \to \mathbb{R}$, $K(\cdot) \not\equiv 0$, we define the weighted Lebesgue space $L^s(\mathbb{R}^N, |K(x)|)$ equipped with the semi-norm

$$\|u\|_{L^s(\mathbb{R}^N,|K(x)|)} = \left(\int_{\mathbb{R}^N} |K(x)||u|^s\,dx\right)^{1/s}. \tag{2.1}$$

If $1 < p < N$ and $a < (N-p)/p$, we define $E = \mathcal{D}_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$ as being the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_E^p = \int_{\mathbb{R}^N} |x|^{-ap}|Du|^p\,dx. \tag{2.2}$$

The following Hardy–Sobolev inequality is due to Caffarelli et al. [15], which is called the Caffarelli–Kohn–Nirenberg inequality. There is a constant $S_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bp_*(b)}|u|^{p_*(b)}\,dx\right)^{p/p_*(b)} \le S_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|Du|^p\,dx \quad \forall u \in C_0^\infty(\mathbb{R}^N) \tag{2.3}$$

where $a \le b \le a+1$, $p_*(b) = Np/(N-p(a+1-b))$. From the approximation argument, it is easy to see that (2.3) holds on E.

The natural functional space to study problem (1.1) is $X = E \cap L^p(\mathbb{R}^N, V)$ with respect to the norm

$$\|u\|_X^p = \int_{\mathbb{R}^N} \left(|x|^{-ap}|Du|^p + V(x)|u|^p\right)\,dx. \tag{2.4}$$

The following compact embedding theorem is an extension of the classical Rellich–Kondrachov compactness theorem due to Xuan [1].

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$. Then the embedding $X \hookrightarrow L^s(\Omega, |x|^{-\alpha})$ is compact if $1 \le s < Np/(N-p)$ and $\alpha/s < 1 + a + N(s^{-1} - p^{-1})$.*

Let $\varphi_R \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \le \varphi_R \le 1$, $\varphi_R(x) = 0$ for $|x| < R$, $\varphi_R(x) = 1$ for $|x| > R+1$ and $|D\varphi_R(x)| \le C$. For any fixed $R > R_0$, we write $u = \varphi_R u + (1-\varphi_R)u$. Denote $B_R^C = \mathbb{R}^N \setminus B_R$. Then we have the following two theorems which are pertinent to our purpose.

Theorem 2.2. *Let $p < s < p^*$ and $K(x) \in L^\infty(B_{R+1} \setminus B_\tau)$ with $0 < \tau < R_0 < R < +\infty$. Suppose*

$$\lim_{x\to 0} |x|^\gamma |K(x)| < +\infty \quad \text{and} \quad \mathcal{M} := \lim_{R\to+\infty} m(R) < +\infty, \tag{2.5}$$

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