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Application of Faber polynomials to the approximate solution of a generalized boundary value problem of linear conjugation in the theory of analytic functions

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A B S T R A C T

In the present paper the method of successive approximations and Faber polynomials are used to derive the approximate solution of a generalized boundary value problem of linear conjugation on the Lyapunov curve. The conditions for the existence and uniqueness of the solution are presented in the L_2 and $H(\alpha)$ spaces.

1. Introduction

Let Γ be a closed Lyapunov curve [1, p. 90] and let $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$, and $c(t)$ be complex-valued functions on Γ fulfilling the Hölder condition with exponent $0 < \alpha \leqslant 1$ (i.e. belonging to the function class $H(\alpha)$). Moreover, let $a_1(t)\overline{a_2(t)} - \overline{b_1(t)}b_2(t) \neq 0$, $\forall t \in \Gamma$. We seek for two analytic functions $\varphi^+(z)$, $z \in D^+$, and $\varphi^-(z)$, $z \in D^-$, satisfying the boundary condition

$$\begin{aligned} &a_1(t)\varphi^+(t) + b_1(t)\overline{\varphi^+(t)} = a_2(t)\varphi^-(t) + b_2(t)\overline{\varphi^-(t)} + c(t), \quad t \in \Gamma, \\ &\varphi^-(\infty) = 0, \end{aligned} \tag{1}$$

where $\varphi^\pm(t)$ are the limit values of the functions $\varphi^\pm(z)$, D^+ is the interior of the curve Γ, and D^- is its exterior.

The special case of this problem for $a_1(t) \equiv 1$, $b_1(t) \equiv 0$, $b_2(t) \equiv 0$ was studied by D. Hilbert [2] and I.I. Privalov [3]. However, their solutions were incomplete. Full and efficient solution of the problem ($a_1(t) \equiv 1$, $b_1(t) \equiv b_2(t) \equiv 0$) was obtained by F.D. Gakhov [4]. It is worth noting that the following form of the problem

$$\varphi^+(t) = a_2\varphi^-(t) + b_2\overline{\varphi^-(t)}, \quad t \in \Gamma,$$

where the coefficients a_2 and b_2 are constant, can be transformed into the form of the so-called transmission problem from the harmonic functions theory (see [5] for details) which was first discussed by N.I. Muskhelishvili in [6]. Next, in the case $a_1(t) \equiv 1$, $b_1(t) \equiv 0$ the problem (1) was formulated by A.I. Markushevich [7] in the form

$$\varphi^+(t) = a_2(t)\varphi^-(t) + b_2(t)\overline{\varphi^-(t)} + c(t), \qquad \varphi^-(\infty) = 0, \quad t \in \Gamma. \tag{2}$$

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Markushevich solved it for the special case $a_2(t) \equiv c(t) \equiv 0$. The next important results on Eq. (2) were obtained by I.N. Vekua [8], B. Bojarski [9], and L.G. Mikhailov [10] because of the application of this boundary value problem to the theory of fused surfaces. More recently, the problem for multiply connected domains has been discussed in [11]. Its history has been presented in [5]. Physical applications include, among others, electrical conductivity, magnetic permeability, and flow in porous media [12].

In the present paper we construct the solution of the boundary problems (1) and (2) using the method of successive approximations and Faber polynomials. These polynomials and their numerous modifications play a fundamental role in the theory of approximation of complex-valued functions (see the monograph by P.K. Suetin [1] and the papers [13–19]). A comprehensive survey can be found in [20]. As a consequence of their usefulness in the approximation theory, Faber polynomials are a very powerful tool appearing in a plethora of numerical methods, e.g. solving the Dirichlet problem in the plane [21], approximate solutions of singular integral equations (cf. [22,23]) and singular integro-differential equations (see [24,25]), etc.

The advantage of the proposed approach is in its efficiency obtained by the application of Faber polynomials for finding the singular integrals appearing in the successive approximations method. The method is proved to be convergent to the solution of the problem. Moreover, the conditions for the existence and uniqueness of the solution are given in two classes of function spaces, namely L_2 and Hölder continuous functions $H(\alpha)$.

The paper is organized as follows. The algorithm of solving the problem (1) by the method of successive approximations and the theorems on the existence and the uniqueness of the solution depending on the index of the problem are presented in Section 2. In Section 3 we discuss the application of Faber polynomials to the calculation of Cauchy-type singular integrals present in each step of the iterative algorithm while an illustrative example showing the usefulness of our method in practice is covered in Section 4.

2. Solution of the boundary value problem

Let us consider the problem (1). Applying the complex conjugation on both sides of this equality we get the system of equations

$$\begin{aligned} a_1(t)\varphi^+(t) + b_1(t)\overline{\varphi^+(t)} &= a_2(t)\varphi^-(t) + b_2(t)\overline{\varphi^-(t)} + c(t), \\ \overline{b_1(t)}\varphi^+(t) + \overline{a_1(t)}\,\overline{\varphi^+(t)} &= \overline{b_2(t)}\varphi^-(t) + \overline{a_2(t)}\,\overline{\varphi^-(t)} + \overline{c(t)}. \end{aligned} \tag{3}$$

Elimination of $\overline{\varphi^-(t)}$ from this system yields

$$\varphi^+(t) = A(t)\varphi^-(t) + B(t)\overline{\varphi^+(t)} + C(t), \quad t \in \Gamma, \qquad \varphi^-(\infty) = 0, \tag{4}$$

where

$$A(t) = \frac{|a_2(t)|^2 - |b_2(t)|^2}{a_1(t)\overline{a_2(t)} - \overline{b_1(t)}b_2(t)}, \qquad B(t) = \frac{\overline{a_1(t)}b_2(t) - b_1(t)\overline{a_2(t)}}{a_1(t)\overline{a_2(t)} - \overline{b_1(t)}b_2(t)},$$
$$C(t) = \frac{\overline{a_2(t)}c(t) - b_2(t)\overline{c(t)}}{a_1(t)\overline{a_2(t)} - \overline{b_1(t)}b_2(t)}.$$

Let $A(t) \neq 0$, $\forall t \in \Gamma$, and let the origin of a coordinate system $\mathbb{C}_z$ belong to the area D^+. Moreover, let $\operatorname{Ind} A(t) = \varkappa \geqslant 0$. Then by [4] the canonical functions $X^+(z),\ z \in D^+$, and $X^-(z),\ z \in D^-$, of the linear conjugation problem

$$X^+(t) = A(t)X^-(t), \quad t \in \Gamma, \tag{5}$$

have the forms

$$X^+(z) = \exp\left(\Gamma^+(z)\right), \quad z \in D^+,$$

and

$$X^-(z) = z^{-\varkappa} \exp\left(\Gamma^-(z)\right), \quad z \in D^-,$$

respectively. Here

$$\Gamma^\pm(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln\left(\tau^{-\varkappa}A(\tau)\right)}{\tau - z}\,d\tau, \quad z \in D^\pm.$$

Using (5) the boundary problem (4) can be rewritten as

$$\frac{\varphi^+(t)}{X^+(t)} - \frac{\varphi^-(t)}{X^-(t)} = G(t)\frac{\overline{\varphi^+(t)}}{\overline{X^+(t)}} + g(t), \quad t \in \Gamma, \tag{6}$$