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# Hyperbolic conservation laws for continuous two-phase flow without mass exchange

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ABSTRACT

It is well known that classic two-phase flow equation systems have complex characteristic roots and, therefore, constitute an ill-posed initial-value problem. Here we suggest that ill-posedness is due to working with two different material derivatives for the phases, which have varying velocities, but employing the same position vector for both operators. There follows an analysis of the conditions required for a global treatment of both phases, but using only one material derivative for both phases, now coherent with only one position vector. Consequently, new global mass- and momentum-conservation equations for a two-phase flow without mass exchange are derived by strictly following the classic Reynolds' transport theorem. The new global mass-conservation equation proposed would only be valid if the 'zero-net-mass-flux' condition, another independent equation, was fulfilled. We also found that the new equation system is well-posed, i.e. its two characteristic roots are real if a new relation between velocities and densities is satisfied. Finally, we have highlighted the strong connections of new conservation laws with classic treatments, and also shown that minor modifications of the current equation system would turn it into a hyperbolic one, thus easing the computational solution of this complex problem.



## 1. Introduction

Two-phase flow is in the core of a large amount of important engineering processes (power, heat transfer, and chemical process industries), and physical processes (geo-meteorological and biological flows). The key characteristic of a two-phase flow is the coexistence of two different phases with different velocity fields in the domain, which highly complicates the solution of the corresponding equation system. The computational fluid dynamics (CFD) simulation of two-phase flow based on the Euler–Euler approach, also called the separated two-fluid model, has much lower computational demands and is the only practical method for simulating large-scale systems. This method usually also includes the hypothesis of a continuum [1–3].

The separated two-fluid model, which was originally proposed by Wallis [4], is considered the 'standard' two-fluid model. It consists of two sets of conservation equations for mass, momentum and energy for the two phases. Although it has demonstrated some success in simulating two-phase flow in pipelines, the separated flow model suffers from an ill-posedness problem. When the relative velocity between the phases exceeds a critical value, the governing equations do not possess real characteristics [3,5,6].

This ill-posedness suggests that the results of the standard two-fluid model do not reflect the real flow physics inside the pipe for these conditions and are incorrect in some essential physical respects. We could also expect their characteristics to become real when their deficiencies are corrected [5,6].

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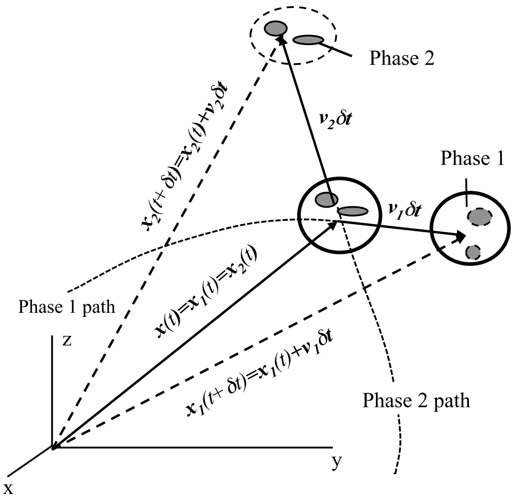


**Fig. 1.** Different position vectors for phase 1 and phase 2.

The main suggestion of this work is that the ill-posedness of the classic equation system for a two-phase flow would be due to working simultaneously with the two classic material derivatives, i.e. $D_1\,()\,/Dt$ following phase 1 and $D_2\,()\,/Dt$ following phase 2, whereas both derivatives are based on the same position vector $\boldsymbol{x}(t)$. Then using both derivatives simultaneously, we would follow different paths of derivation (corresponding to the two different velocities), although through the same position vector, which is clearly impossible.

To centre the problem, we briefly review the classic individual mass-conservation equations for a two-phase flow without mass exchange between the phases [1–3], namely

$$\frac{D_1 m_1}{Dt} = 0 \Rightarrow \frac{\partial \rho_1 \alpha_1}{\partial t} + \nabla \cdot (\rho_1 \alpha_1 \boldsymbol{v_1}) = 0, \tag{1}$$

$$\frac{D_2 m_2}{Dt} = 0 \Rightarrow \frac{\partial \rho_2 \alpha_2}{\partial t} + \nabla \cdot (\rho_2 \alpha_2 \boldsymbol{v_2}) = 0, \tag{2}$$

in which $\boldsymbol{v}$ is the velocity vector, $\rho$ is the material density and $\alpha$ is the void fraction, and subscripts 1 and 2 stand for phase 1 and phase 2, respectively. The physical key point of a two-phase flow, i.e. that the phase velocities are different from each other, is also assumed throughout this work.

Based on the classic material derivative concept, Eq. (1) states that a mass system, composed exclusively of phase 1, will be conserved while it is moving and deforming through time and space. The same is affirmed for a mass system exclusively composed of phase 2 in Eq. (2).

Notice that both equations treated individually are essentially correct. Nevertheless, if we merely add Eqs. (1)–(2), the above-commented discrepancy concerning the impossibility of using the same position vector for both phases arises.

This idea is sketched in Fig. 1, which defines a mass differential system composed of phase 1 and phase 2 at some "point" of the flow field in time $t$. At time $t$, the different phase elements do coincide at the same "point" $\boldsymbol{x}\,(t)$. Thus, their respective position vectors also exactly agree, i.e. $\boldsymbol{x_1}\,(t) = \boldsymbol{x_2}\,(t) = \boldsymbol{x}\,(t)$. However, after a small time interval $\delta t$, $m_1$ will have moved to a new position defined by the vector $\boldsymbol{x}\,(t) + \boldsymbol{v_1}\delta t$ whereas $m_2$ will have moved to a completely different location now defined by a different vector $\boldsymbol{x}\,(t) + \boldsymbol{v_2}\delta t$.

It seems clear that, although the phase position vectors at time $t$ coincide, after a small time interval their respective position vectors are completely different,

$$\boldsymbol{x_1}\,(t+\delta t) = \boldsymbol{x}\,(t) + \boldsymbol{v_1}\delta t \neq \boldsymbol{x_2}\,(t+\delta t) = \boldsymbol{x}\,(t) + \boldsymbol{v_2}\delta t. \tag{3}$$

Therefore, using the two former material derivatives simultaneously is clearly incoherent with working with only one position vector.

Notice that although standard treatments of two-phase flow definitely accept different material derivatives for the phases, the position vector used is the same for both phases [1–3].

Finally, this suggested discrepancy is also highlighted on comparing the calculation of the phase velocities by applying the classic phase material derivative operators [1–3] to the same position vector with the strict definition of the velocity of a mass element, i.e. the time rate of change of its position vector [7–9],

$$\frac{D_1\boldsymbol{x}}{Dt} = \frac{\partial\,\boldsymbol{x}}{\partial t} + u_1\frac{\partial\,\boldsymbol{x}}{\partial x} + v_1\frac{\partial\,\boldsymbol{x}}{\partial y} + w_1\frac{\partial\,\boldsymbol{x}}{\partial z} = \boldsymbol{v_1} = \lim_{\delta t\to 0}\frac{\boldsymbol{x}\,(t+\partial t) - \boldsymbol{x}\,(t)}{\delta\,t}, \tag{4}$$

$$\frac{D_2\boldsymbol{x}}{Dt} = \frac{\partial\,\boldsymbol{x}}{\partial t} + u_2\frac{\partial\,\boldsymbol{x}}{\partial x} + v_2\frac{\partial\,\boldsymbol{x}}{\partial y} + w_2\frac{\partial\,\boldsymbol{x}}{\partial z} = \boldsymbol{v}_2 = \lim_{\delta t\to 0}\frac{\boldsymbol{x}\,(t+\partial t) - \boldsymbol{x}\,(t)}{\delta t}. \tag{5}$$