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A robust numerical method for a control problem involving singularly perturbed equations☆

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ABSTRACT

We consider an unconstrained linear–quadratic optimal control problem governed by a singularly perturbed convection–reaction–diffusion equation. We discretize the optimality system by using standard piecewise bilinear finite elements on the graded meshes introduced by Durán and Lombardi in (Duŕan and Lombardi 2005, 2006). We prove convergence of this scheme. In addition, when the state equation is a singularly perturbed reaction–diffusion equation, we derive quasi-optimal a priori error estimates for the approximation error of the optimal variables on anisotropic meshes. We present several numerical experiments when the state equation is both a reaction–diffusion and a convection–reaction–diffusion equation. These numerical experiments reveal a competitive performance of the proposed solution technique.

1. Introduction

We are interested in the study of new and efficient solution techniques for an unconstrained linear–quadratic optimal control problem involving a convection–reaction–diffusion equation. To be precise, let $\Omega = (0, 1)^2$. Given a desired state $\mathsf{y}_d : \Omega \to \mathbb{R}$, we define the cost functional

$$J(\mathsf{y}, \mathsf{u}) = \frac{1}{2}\|\mathsf{y} - \mathsf{y}_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\mathsf{u}\|^2_{L^2(\Omega)}, \tag{1.1}$$

where $\lambda > 0$ denotes the so-called regularization parameter. Let $\mathsf{c}, \mathsf{f} : \Omega \to \mathbb{R}$ be fixed functions and $\mathbf{b} : \Omega \to \mathbb{R}^2$ be a given vector field. We shall be concerned with the following optimal control problem: Find

$$\min J(\mathsf{y}, \mathsf{u}) \tag{1.2}$$

subject to the singularly perturbed convection–reaction–diffusion equation

$$-\varepsilon^2 \Delta \mathsf{y} + \mathbf{b} \cdot \nabla \mathsf{y} + \mathsf{c}\mathsf{y} = \mathsf{u} + \mathsf{f} \quad \text{in } \Omega, \qquad \mathsf{y} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where ε denotes the *perturbation parameter* and satisfies $0 < \varepsilon \ll 1$. The term u denotes the *control variable*, and y, the solution to the *state equation* (1.3), corresponds to the *state variable*. We will also be interested in the particular scenario

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where $\mathbf{b} \equiv 0$, which yields a singularly perturbed reaction–diffusion problem as state equation:

$$-\varepsilon^2 \Delta \mathsf{y} + \mathsf{c}\mathsf{y} = \mathsf{u} + \mathsf{f} \quad \text{in } \Omega, \qquad \mathsf{y} = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Since the optimal control problem (1.2)–(1.3) does not involve control constraints, the associated optimality system corresponds to a coupled one involving two singularly perturbed convection–reaction–diffusion equations: the state and adjoint equations; see [1–4]. The adjoint equation has a convection component that is the negative of the one appearing in (1.3) [2,3]. This leads to the computational challenge of how to efficiently resolve the optimality system associated with (1.2)–(1.3), which in turn demands the use of an efficient method to solve the state equation (1.3). When solving the latter, it is known that standard finite element techniques lead to strongly oscillatory solutions unless the mesh-size is sufficiently small with respect to the ratio between ε and $\|\mathbf{b}\|$. In addition, the sharp boundary and interior layers and corner and edge singularities, that usually appear in the solution to these types of problems, must be efficiently resolved [5–8].

In the context of optimal control problems, to overcome such difficulties, different stabilized finite element techniques have been proposed and analyzed in the literature; see [9,1,2,10–14]. To the best of our knowledge, the first work that analyzed a stabilized scheme is [2]. This work considers the streamline upwind/Petrov Galerkin (SUPG) method, elaborates on the fact that the optimize-then-discretize and discretize-then-optimize approaches do not coincide and explores their respective advantages. Later, local projection stabilization (LPS) techniques were proposed in [1]. These techniques have the advantage that, due to the symmetry of the proposed stabilization term, optimize-then-discretize and discretize-then-optimize coincide. In addition, the authors derive global a priori error estimates for the approximation error of the optimal variables. However, it is important to notice, as pointed out in [3], that these global error estimates (see also [1,15,11–13]) contain constants that depend on derivatives of the optimal variables and so in the presence of interior and boundary layers such estimates become meaningless. This motivates the local a priori error analysis of [3], where the SUPG method is used to approximate the solution of the state equation (1.3). The authors derive local a priori error estimates in subdomains $\Omega_0 \subset \Omega$ that do not include boundary or interior layers. However, the presence of boundary layers pollutes the numerical solution, even in subregions where the solution is smooth. This is due to the fact that the boundary layers are not sufficiently resolved; see [3] for a discussion.

In some particular cases, where some information on the behavior of the solution to the single state equation (1.3) or (1.4) is available, it is possible to design a priori graded meshes to efficiently approximate the boundary layers exhibited by the solution; see [16–18,8]. Some well-known approximation techniques of this kind are those based on the so-called Shishkin meshes [19,20,8]. In the context of the optimal control theory, some numerical experiments provided in [3, Section 5.2] show promising results when Shishkin meshes are considered. In [21], the one dimensional version of (1.2)–(1.3) is analyzed and discretized using these meshes. The authors derive optimal rates of convergence in the energy-norm, but the presented error estimates for the L^2-norm are not optimal in terms of approximation.

In this work we propose a different approach based on the finite element approximation of the optimality system, associated with the optimal control problem (1.2)–(1.3), on the graded or anisotropic meshes proposed by Durán and Lombardi in [16,17]. In the case that $\mathbf{b} = 0$, i.e., when (1.4) corresponds to the state equation, we propose a solution technique and derive a priori quasi-optimal error estimates in both the energy and the L^2-norm. The error estimates derived in the energy-norm are ε-independent [22,23], which is a property that Shishkin meshes do not satisfy. We also propose an approximation scheme for solving the optimal control problem (1.2)–(1.3), where the state equation (1.3) contains a non-zero constant vector field $\mathbf{b}$. We prove the convergence of the scheme by invoking the theory of Γ-convergence. We design several computational experiments that show a competitive performance of the proposed solution technique when it is compared with adaptive stabilized schemes. In addition, we observe that

- the experimental rates of convergence in both the energy and the L^2-norm are quasi-optimal in terms of approximation.
- the pollution effect discussed in [3] is not observed. This is due to the fact that the boundary layers are appropriately approximated.

We comment that, the numerical analysis of the proposed approximation scheme to solve (1.2)–(1.3), where $\mathbf{b}$ is a non-zero and constant vector field, could be derived under strong assumptions on the optimal control variable; see Section 4.2.1 for a discussion. The main disadvantage of our graded-mesh scheme is that it is based on the anisotropic error estimates analyzed in [16] which are valid under a tensor product structure of the domain Ω.

The outline of this paper is as follows. In Section 2 we introduce the functional framework that is suitable for analyzing the optimal control problem (1.2)–(1.3) and state existence and uniqueness results in conjunction with optimality conditions. Section 3 is a review of the graded finite element techniques developed in [17,16] to solve the state equations (1.3) and (1.4), respectively. In Section 4 we propose a numerical technique to solve the optimal control problem (1.2)–(1.3): a fully discrete scheme that discretizes the optimal control and state with standard piecewise bilinear finite elements on anisotropic meshes. Section 4.1 contains a complete quasi-optimal a priori error analysis, in both the energy and the L^2-norm, when (1.4) is considered as the state equation. In Section 4.2, we assume that $\mathbf{b}$ is a non-zero vector field and prove the convergence of the proposed scheme via the theory of Γ-convergence. Finally, in Section 5 we present several numerical experiments that reveal a competitive performance of the proposed solution technique. We also explore computationally the performance of our method when solving a constrained optimal control problem with (1.3) as the state equation.

Throughout the manuscript, the relation $a \lesssim b$ indicates that $a \leq Cb$, with a constant C that does not depend on either a or b. The value of C might change at each occurrence.