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Neural network solution for an inverse problem associated with the Dirichlet eigenvalues of the anisotropic Laplace operator

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ABSTRACT

An innovative numerical method based on an artificial neural network is presented in order to solve an inverse problem associated with the calculation of the Dirichlet eigenvalues of the anisotropic Laplace operator. Using a set of predefined eigenvalues obtained by solving repeatedly the direct problem, a radial basis neural network is designed with the purpose to find the appropriate components of the anisotropy matrix, related to the Laplace operator, and thus solving the associated inverse problem. The finite element method is used to solve the direct problem and to create the training set for the first radial basis neural network. A nonlinear map of the Dirichlet eigenvalues as a function of the anisotropy matrix is then obtained. This nonlinear relationship is later inverted and refined, by training a second radial basis neural network, solving the aforementioned inverse problem. Some numerical examples are presented to prove the effectiveness of the introduced method.

1. Introduction

At sufficiently small scales, any material exhibits anisotropic properties which manifest themselves differently, depending on the physical method used to probe its microscopic structure; either these are X-rays, radar signals, electromagnetic or mechanical waves. These anisotropies are responsible for a great number of nonlinear and observable physical phenomena. The study of anisotropies is important, for example to understand material rupture and changes in the transmission wave front in radiation applications. Understanding anisotropies is critical in several research areas such as those associated to granular media and graphene. Other applications where anisotropy plays a fundamental role include wave structure interactions and antenna development.

For example in order to characterize the behavior of new materials under different stress tests, one needs to know their acoustic, electromagnetic and/or elastic properties, or to have an optimal method for their estimation. These properties encode implicitly the anisotropies, and a poor understanding of these can cause unwanted behavior under different laboratory tests. Similarly, electromagnetic and mechanical properties of materials such as density, dielectric permittivity, electric conductivity, thermal conductivity, or heat dissipation depend heavily on the process that created the material and, in some cases, on the origin of the anisotropies.

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In acoustics, the anisotropy can be related to variations in the geometry of the studied domain. The anisotropic Laplace–Beltrami operator is employed to analyze and characterize shapes through a spectral analysis in acoustic domains: see for example Reuter et al. [1] and Andreux et al. [2]. The acoustic eigenvalues can be obtained directly using the Finite Element Method (FEM). The calculation of eigenvalues using a FEM is not a new technique. The literature on the area is numerous and describes a number of valuable models in order to solve spectral problems. Some contributions include the works of Babuška and Osborn [3], Oden and Reddy [4], and Kolata [5]. Also, hybrid methods (hybrid between Boundary Element Method (BEM) and FEM) are mentioned in Banerjee et al. [6], Coyette and Fyfe [7] and Ali et al. [8]. More efficient techniques are described in Kirkup and Amini [9], Durán et al. [10], and in related works of Chen et al. [11,12].

On the other hand, numerical solutions for the inverse problem, i.e., calculating numerically the components of the anisotropy matrix through the eigenvalues has been a less explored area, due to the difficulty to recover the geometry only from the knowledge of the complete set of acoustic eigenvalues. The possibility to recover the geometry through the eigenvalues has been thoroughly analyzed in Kac [13]. Gordon et al. [14] show that it is possible to produce the same eigenvalues using different geometries. The construction of these isospectral geometries proved that it is impossible to use the eigenvalues as the only tool for shape differentiation. However, depending on the boundary conditions and/or depending of the range of variability of the geometry, the eigenvalues allow to characterize an object. In Ossandon et al. [15], a method based on BEM is described, in which it is possible to use the eigenvalues for non-destructive shape evaluation. In the work of Payne et al. [16], a variety of universal inequalities are presented for the eigenvalue problem, which do not depend explicitly on the shape or size of the underlying domain. In Ashbaugh and Benguria [17], an upper limit ratio between the eigenvalues is established, and the stability for these inequalities is proven in Melas [18]. These results indicate that it is possible to derive the shape of a domain if this is convex and if the ratio between the first and second eigenvalues is close to those encountered on a disk.

The idea to use Artificial Neural Networks (ANN) to solve eigenvalue problems appears in different contexts. In the case of matrices (e.g. Liu et al. [19]), it is possible to use a neural network to compute the largest and smallest eigenvalues of a real symmetric matrix. Also in Song and Yam [20], a recurrent complex neural network is used to calculate the inverse of a matrix in real time. In Georgiou [21], the numerical range of a normal matrix is analyzed through a neural network where all found eigenpairs can be used as stored memories in order to find other eigenpairs. In Saliah and Lowther [22], neural networks are used to model the magnetic behavior of anisotropic materials.

However, there is scarce literature in which ANN have been applied to solve the inverse eigenvalue problem of an operator, which is the case this article addresses. An effort in this direction was made by Ossandon et al. [23] for the elasticity operator, where the authors show that ANN can be used to characterize the Lamé coefficients of an elastic material through the eigenvalues, with a lower computational cost than methods based on FEM. Thus, the use of ANN in order to characterize material properties through the eigenvalues, seems to be a very attractive method for reducing the computational time involved and for the calculation of specific anisotropy features. Let us notice that depending on the field under study, many different ANN models are proposed. In this study, the type of neural network chosen is the feedforward radial basis neural network with backpropagation algorithm, due to its robustness and generalization capacity (see [24]).

Let us mention that several physical applications are modeled mathematically using the compact differential operator $-\mathrm{div}(\mathbf{A}(\cdot)\cdot\nabla)$, where $\mathbf{A} = [a_{ij}]_{ij} \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^{n\times n})$ is a matrix-valued function and $\Omega \subseteq \mathbb{R}^n$ is an open bounded set and Γ its boundary. For example, in heat conduction, an anisotropic and inhomogeneous media, where the thermal conductivity is not the same in all directions, is characterized by a symmetric matrix-valued function $\mathbf{A}(\cdot)$ (not proportional to the identity matrix). In electrostatics, the matrix-valued function $\mathbf{A}(\cdot)$ could model an anisotropic and inhomogeneous behavior of the permittivity associated to a dielectric medium. Usually, in order to ensure the well-posedness of the following elliptic boundary value problem (for a given $f \in L^2(\Omega) := \{u : \Omega \to \mathbb{R} : \int_\Omega |u|^2 d\mathbf{x} < +\infty\}$)

$$\begin{cases} -\mathrm{div}(\mathbf{A}(\cdot)\cdot\nabla)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

the following properties are demanded to the matrix-valued function $\mathbf{A}(\cdot)$:

$$\begin{aligned} &\text{(i) } 1 \leqslant \forall i, j \leqslant n, \quad \text{and} \quad \forall \mathbf{x} \in \overline{\Omega}, \quad a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \\ &\text{(ii) } \exists \nu > 0 \quad \text{such that } \forall \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x}^T \cdot \mathbf{A}(\mathbf{x}) \cdot \mathbf{x} \geqslant \nu \|\mathbf{x}\|^2, \end{aligned} \tag{2}$$

where (i) implies that $-\mathrm{div}(\mathbf{A}(\cdot)\cdot\nabla)$ is self-adjoint, which means that if there exists a point spectrum of this operator then this point spectrum is contained in a compact subset of the real axis. On the other hand, (ii) is well known as the uniform ellipticity property of $\mathbf{A}(\cdot)$ and implies the injectivity of the differential operator giving the uniqueness of the solution of (1).

In this work, we propose the use of a numerical method based on ANN, in order to calculate the components of the anisotropy matrix $\mathbf{A}$ associated with the compact differential operator $-\mathrm{div}(\mathbf{A}\cdot\nabla)$ through its eigenvalues. For this, we assume in the following that $\mathbf{A}$ is a constant matrix ($\mathbf{A}$ does not depend on $\mathbf{x} \in \overline{\Omega}$), i.e. we consider a homogeneous anisotropic material. See Fig. 1.

The paper is organized as follows: In Section 2 the mathematical formulations of the direct and inverse problems associated with the calculation of the Dirichlet eigenvalues of the anisotropic Laplace operator are presented. Section 3 describes the methodology, using a FEM technique, in order to obtain a solution of the direct problem. Section 4 shows the