Computers and Mathematics with Applications 72 (2016) 1164–1176

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Convergence analysis of some iterative methods for a nonlinear matrix equation

Vejdi I. Hasanov [a,*], Sevdzhan A. Hakkaev [b]

[a] Konstantin Preslavsky University of Shumen, Faculty of Mathematics and Informatics, Shumen, Bulgaria
[b] Faculty of Arts and Sciences, Department of Mathematics and Computer Science, Istanbul Aydin University, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 14 October 2015
Received in revised form 19 May 2016
Accepted 25 June 2016
Available online 16 July 2016

Keywords:
Nonlinear matrix equation
Fixed point iteration
Inversion free iteration
Newton method
Stein iteration
Convergence rate

ABSTRACT

We study iterative methods for finding the largest Hermitian positive definite solution of the matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$. Convergence rates of the basic fixed point iteration, inversion free variants of the basic fixed point iteration, and Stein iteration are considered. Some numerical examples are presented to illustrate the convergence behaviour of the various algorithms.

1. Introduction

In this paper we consider the matrix equation

$$X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q, \tag{1}$$

where $A_1, A_2, \ldots, A_m$, are $n \times n$ complex matrices, Q is a Hermitian positive definite matrix and X is unknown matrix. With A^* we denote the conjugate transpose of the matrix A.

Eq. (1) can be written in form

$$Y + \sum_{i=1}^{m} B_i^* Y^{-1} B_i = I, \tag{2}$$

where $Y = Q^{-1/2} X Q^{-1/2}$, $B_i = Q^{-1/2} A_i Q^{-1/2}$ and $Q^{1/2}$ is a positive definite square root of Q. Therefore, Eq. (1) is solvable if and only if the Eq. (2) is solvable. When $m = 1$ Eq. (1) has many applications: in ladder network, dynamics programming, stochastic filtering [1], control theory [2] and so on. For $m > 1$ Eq. (1) is introduced by He and Long [3]. Bini et al. [4] considered the equation

$$X + \sum_{i=1}^{m} C_i X^{-1} D_i = E \tag{3}$$

* Corresponding author.
E-mail addresses: v.hasanov@fmi.shu-bg.net (V.I. Hasanov), shakkaev@fmi.shu-bg.net (S.A. Hakkaev).

http://dx.doi.org/10.1016/j.camwa.2016.06.035

arising in Tree-Like stochastic processes, where $C_i, D_i, i = 1, 2, \ldots, m$ and E are $n \times n$ given real matrices with the properties: $E = B - I$ and B is sub-stochastic; C_i and D_i have nonnegative elements; and the matrices $I + E + D_i + C_1 + \cdots + C_m, \ i = 1, 2, \ldots, m$ are stochastic.

In this paper, we write $A > 0$ ($A \geq 0$) if A is a Hermitian positive definite (semidefinite) matrix. For Hermitian matrices A and B, we write $A > B$ ($A \geq B$) if $A - B > 0$ ($A - B \geq 0$). A Hermitian solution X_L of a matrix equation is called largest if $X_L \geq X$ for any Hermitian solution X of the equation.

Eq. (1) for $m = 1$ has been studied extensively in the last two decades, and the result mainly concern on the following: conditions for existing and properties of a positive definite solution [1,2,5]; iterative methods for computing of a positive definite solution [1,2,5–8] and the perturbation bounds of the largest positive definite solution [9,10]. Long et al. [11] studied Eq. (1) for $m = 2$. They obtained some conditions on the matrix coefficients for the existence of the positive definite solution and considered a basic fixed point iteration and an inversion free variant. Vaezzadeh et al. [12] considered inversion free iterative methods for (1) when $m = 2$, also. They give some generalization of the convergence theorems of Guo and Lancaster [7]. Hasanov and Ali [13] improved the results of Vaezzadeh et al. in [12]. Popchev et al. [14,15] made a perturbation analysis of (1) for $m = 2$.

In the general case, Eq. (1) was considered by He and Long [3]. They proposed a basic fixed point iteration and an inversion free variant method for finding the largest positive definite solution of (1). But the problem of convergence rate in [3] was not considered. Hasanov and Hakkaev [16] considered the Newton's method for Eq. (1). Duan et al. [17] derived the perturbation bound for the largest positive definite solution of (1) based on the matrix differentiation. Moreover, many authors have been investigated similar or more general nonlinear matrix equations $X + A^*\mathcal{F}(X)A = Q$ [18,19], $X \pm \sum_{i=1}^{m} A_i^*\mathcal{F}(X)A_i = Q$ [20], $X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q$ $(0 < q \leq 1)$ [21] and $C + \sum_{i=1}^{m} \sigma_i A_i^* X^{p_i} A_i = 0, \ \sigma_i = \pm 1$ [22,23].

Motivated by the work in [3,4,11–23] we continue to study iterative methods for solving the Eq. (1). The paper is organized as follows: in Section 2, we study the convergence rate of the basic fixed point iteration and its inversion free variants for computation of the largest positive definite solution of (1). In Section 3, we propose a new iterative method by modification of the Newton's method and present a method of Bini et al. [4]. In Section 4, we give some numerical experiments.

Throughout the paper, the symbols $\|A\|$ and $\rho(A)$ denote the spectral norm and the spectral radius of a matrix A, respectively. For two matrices A and B, $A \otimes B$ denotes the Kronecker product of A and B. $\mathbf{C}^{n\times n}$ denotes the set of all $n \times n$ matrices, $\mathcal{H}^n$ the set of all $n \times n$ Hermitian matrices, $\mathcal{H}_+^n$ (int $\mathcal{H}_+^n$) the set of all $n \times n$ positive semidefinite (definite) matrices. $\mathcal{H}^n$ is Banach space and Hilbert space endowed with the inner product $\langle A, B \rangle = \mathrm{tr} A^* B$ and the Frobenius norm $\|A\|_F = \langle A, A \rangle^{1/2}$. We use $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$, also.

2. Convergence analysis of basic fixed point iteration and its inversion free variants

In this section we generalize the result of Guo and Lancaster [7] for $m = 1$, i.e., we give convergence rates of the following iterative methods:

Basic fixed point iteration (BFPI)

$$\begin{cases} X_0 = Q, \\ X_{k+1} = Q - \displaystyle\sum_{i=1}^{m} A_i^* X_k^{-1} A_i, \quad k = 0, 1, \ldots. \end{cases} \tag{4}$$

First inversion free variant of BFPI (FIFV-BFPI)

$$\begin{cases} X_0 = Q, \quad 0 < Y_0 \leq Q^{-1}, \\ X_{k+1} = Q - \displaystyle\sum_{i=1}^{m} A_i^* Y_k A_i, \\ Y_{k+1} = Y_k(2I - X_k Y_k), \qquad k = 0, 1, \ldots. \end{cases} \tag{5}$$

Second inversion free variant of BFPI (SIFV-BFPI)

$$\begin{cases} 0 < Y_0 \leq Q^{-1}, \quad X_0 = Q - \displaystyle\sum_{i=1}^{m} A_i^* Y_0 A_i, \\ Y_{k+1} = Y_k(2I - X_k Y_k), \\ X_{k+1} = Q - \displaystyle\sum_{i=1}^{m} A_i^* Y_{k+1} A_i, \qquad k = 0, 1, \ldots. \end{cases} \tag{6}$$

In [3] He and Long proved that: if Eq. (1) has a positive definite solution, then there exist a largest positive definite solution X_L of (1), and the *BFPI* (4) defines a monotone decreasing and bounded matrix sequence, which converges to X_L [3, Theorem 2.3].