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A class of preconditioned generalized local PSS iteration methods for non-Hermitian saddle point problems

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ABSTRACT

In this paper, a class of new methods based on the positive-definite and skew-Hermitian splitting scheme, called preconditioned generalized local positive-definite and skew-Hermitian splitting (PGLPSS) methods, are considered to solve non-Hermitian saddle point problems. The convergence properties of the proposed methods are analyzed, which prove that the PGLPSS methods are convergent if the iteration parameters and parameter matrices satisfy appropriate conditions. Some numerical experiments are provided to verify the efficiency of the proposed method, showing the competitiveness and efficiency of this novel method over other testing methods, whether it served as a preconditioned iteration method or as a preconditioner to the Krylov subspace method.

1. Introduction

We consider the iteration solution of large sparse non-Hermitian saddle point problem of the form

$$\mathscr{G}\mathrm{x} = \begin{pmatrix} G & E \\ -E^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1.1}$$

where $G \in \mathbb{C}^{n\times n}$ is non-Hermitian matrix and its Hermitian part $H = \frac{1}{2}(G + G^*)$ is positive definite matrix, $E \in \mathbb{C}^{n\times m}$ is a matrix of full column rank, $x, p \in \mathbb{C}^n$ and $y,\ q \in \mathbb{C}^m$ are given vectors with $n \geq m$, and E^* denotes the conjugate transpose of the matrix E. It readily demonstrates that the solution of (1.1) exists and is unique under the assumptions above; see [1–4]. The linear systems (1.1) arise in many scientific computing and engineering applications such as constrained optimization, computational fluid dynamics, mixed finite element methods for solving elliptic partial differential equations and Stokes problems, constrained least-squares problems, structure analysis and so on; see [5,6]. For such linear systems, a large number of popular and effective iteration methods have been studied in the literatures, such as Uzawa-type methods [7,8], preconditioned Krylov subspace iteration methods [9–14], Hermitian and skew-Hermitian splitting (HSS) iteration methods [15–19], and restrictively preconditioned conjugate gradient methods [20–22]. For more details one can refer to [23].

In all the methods stated above, the HSS method has attracted many researchers' attention due to its promising performance and elegant mathematical properties. Based on the thought of the HSS iteration, many other effectual iteration methods are devised for solving non-Hermitian (positive-definite or positive semi-definite) systems of linear equations, such as normal and skew-Hermitian splitting (NSS) method [24,25], positive-definite and skew-Hermitian splitting (PSS)

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method [26], inexact HSS method [27], relaxed HSS method [28], generalized HSS (GHSS) method [29], modified GHSS method [30], asymmetric and skew-Hermitian splitting method [31], generalized PSS method [32], local HSS (LHSS) method or modified local HSS (MLHSS) method [33], generalized LHSS (GLHSS) and preconditioned GLHSS (PGLHSS) iteration method [34,35], modified relaxed splitting (MRS) method [36] and so on. These iteration methods can not only be used as stationary iteration solvers, but they can also serve as preconditioners for Krylov subspace methods [13], leading to a more efficacious class of solvers.

Notice that the aforementioned iteration methods are all derived or inspired by HSS method. However, it is important to note that the PSS method possesses its special advantages since some special cases of PSS method such as block triangular (or triangular) and skew-Hermitian splitting (BTSS or TSS) methods may save considerably computational costs than HSS (-like) iteration method. Specifically, the BTSS iteration method can solve both Hermitian and strongly non-Hermitian positive-definite systems of linear equations more effectively than HSS iteration method. Recently, based on the idea of the relaxed HSS iteration [28] and the PSS iteration method, a relaxed deteriorated PSS iteration method [37] is presented for solving saddle point problems from the Navier–Stokes equation and an alternating positive semi-definite splitting iteration method [38] is studied for solving non-Hermitian saddle point problems from time-harmonic eddy current models. Besides, a class of Uzawa-PSS iteration methods based on the Uzawa iteration scheme and the PSS iteration method are proposed in [39].

Inspired by this, in this paper, a new class of preconditioned generalized local positive-definite and skew-Hermitian splitting (PGLPSS) iteration methods are proposed for solving the non-Hermitian saddle point problems. The convergence analysis of the PGLPSS iteration method is presented and the convergence conditions are derived. To evaluate the effectiveness and the feasibility of the present method, some numerical comparisons are given and experiment results show that the new proposed method is superior to some of existing methods, i.e., the PGLHSS method, the MLHSS method and the HSS method whether it served as a preconditioned iteration method or as a preconditioner to the Krylov subspace method.

This paper is organized as follows. Section 2 introduces the new proposed method, i.e., preconditioned generalized local positive-definite and skew-Hermitian splitting (PGLPSS) method. Section 3 substantiates theoretically that the PGLPSS iteration method in solving the non-Hermitian saddle point problem (1.1) is convergent. Numerical experiments are performed and the numerical comparisons are given in Section 4, which showed that this novel method has great superiority in some degree compared with some testing methods for solving the non-Hermitian saddle point problems. Finally, Section 5 concludes this paper with some remarks.

2. The preconditioned generalized local PSS method

The main purpose of this section is to introduce the new preconditioned generalized local positive-definite and skew-Hermitian splitting method. Before doing this, let us use the following parameterized block-diagonal preconditioner

$$\mathscr{P} = \begin{pmatrix} \tau_1 P_1 & 0 \\ 0 & \tau_2 P_2 \end{pmatrix} \tag{2.1}$$

to precondition the original system (1.1) from the left, where $P_1 \in \mathbb{C}^{n\times n}$ and $P_2 \in \mathbb{C}^{m\times m}$ are prescribed Hermitian positive definite matrices with τ_1, τ_2 being positive numbers. Unless otherwise specified, we always assume that $P_1 G = G P_1$ holds, then we can obtain the following equivalently linear system

$$\begin{pmatrix} \tau_1 P_1 & 0 \\ 0 & \tau_2 P_2 \end{pmatrix} \begin{pmatrix} G & E \\ -E^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tau_1 P_1 & 0 \\ 0 & \tau_2 P_2 \end{pmatrix} \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2.2}$$

namely,

$$\begin{pmatrix} \tau_1 P_1 G & \tau_1 P_1 E \\ -\tau_2 P_2 E^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tau_1 P_1 p \\ -\tau_2 P_2 q \end{pmatrix}. \tag{2.3}$$

We remark that similar preconditioning strategies in (2.2)–(2.3) have been exploited by Bai et al. in [19]. If we denote above coefficient matrix as below

$$\mathscr{G} = \begin{pmatrix} \tau_1 P_1 G & \tau_1 P_1 E \\ -\tau_2 P_2 E^* & 0 \end{pmatrix}. \tag{2.4}$$

Analogously to [40,35], we consider the following splitting $\mathscr{G} = \mathscr{R} - \mathscr{S}$ of (2.4),

$$\mathscr{R} = \begin{pmatrix} Q_1 + P & 0 \\ Q_3 - \tau_2 P_2 E^* & Q_2 \end{pmatrix} \quad \text{and} \quad \mathscr{S} = \begin{pmatrix} Q_1 - S & -\tau_1 P_1 E \\ Q_3 & Q_2 \end{pmatrix}, \tag{2.5}$$

where P and S in (2.5) are the positive-definite and skew-Hermitian splitting of $\tau_1 P_1 G$, i.e., $\tau_1 P_1 G = P + S$, respectively; the matrix $Q_1 \in \mathbb{C}^{n\times n}$ is a Hermitian positive semi-definite matrix, $Q_2 \in \mathbb{C}^{m\times m}$ is a Hermitian positive definite matrix, and $Q_3 \in \mathbb{C}^{m\times n}$ is an arbitrary matrix.