On some fractional generalizations of the Laguerre polynomials and the Kummer function

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ABSTRACT

This paper refers to some generalizations of the classical Laguerre polynomials. By means of the Riemann–Liouville operator of fractional calculus and Rodrigues' type representation formula of fractional order, the Laguerre functions are derived and some of their properties are given and compared with the corresponding properties of the classical Laguerre polynomials. Further generalizations of the Laguerre functions are introduced as a solution of a fractional version of the classical Laguerre differential equation. Likewise, a generalization of the Kummer function is introduced as a solution of a fractional version of the Kummer differential equation. The Laguerre polynomials and functions are presented as special cases of the generalized Laguerre and Kummer functions. The relation between the Laguerre polynomials and the Kummer function is extended to their fractional counterparts.

1. Preliminaries

Fractional calculus is one of the most intensively developing areas of mathematical analysis. Its fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics. On behalf of the nature of their definition the fractional derivatives provide an excellent instrument for the modeling of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical methods [1]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [2,3] and, recently, of the temperature field problem in oil strata [4]. In special treaties (as in [5–8]) the mathematical aspects and applications of the fractional calculus are extensively discussed.

In this paper the Laguerre functions are derived by Rodrigues' type representation formula for the classical Laguerre polynomials, generalized by means of the Riemann–Liouville fractional differentiation operator. The generalized Kummer function is obtained by a modified power series method as a solution of the fractional extension of the Kummer differential equation. By a similar approach the generalized Laguerre function is introduced.

For our purposes we adopt in this paper the **Riemann–Liouville fractional derivative** of $f(t)$ of order μ, defined by

$$D^\mu f(t) \equiv D^m \left[J^{m-\mu} f(t) \right],$$

where $m \in \mathbb{N}, m - 1 \le \mu < m$, and

$$J^{m-\mu} f(t) \equiv \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{m-\mu-1} f(\tau) \mathrm{d}\tau$$

is the Riemann–Liouville fractional integral of $f(t)$ of order $m - \mu$.

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E-mail addresses: stefan.mirevski@gmail.com (S.P. Mirevski), BoyadjievL@yahoo.com (L. Boyadjiev).
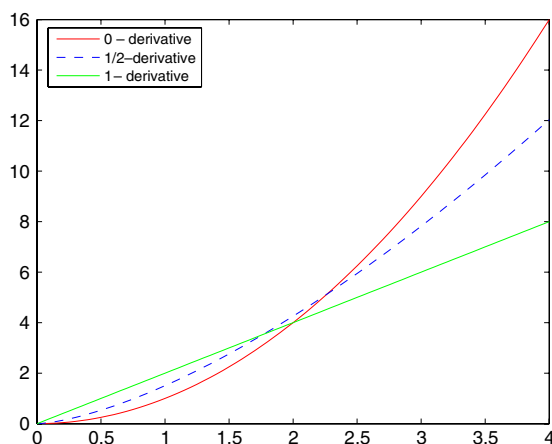
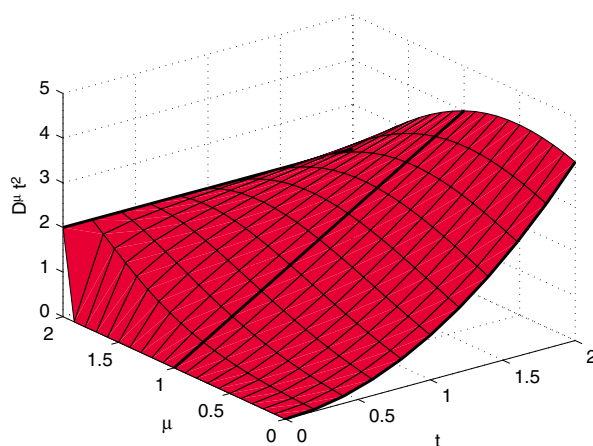**Fig. 1.** 1/2-fractional derivative of $f(t) = t^2$.

Fig. 2. Fractional derivatives of $f(t) = t^2$ of order $0 \leq \mu \leq 2$ on the interval $[0, 2]$.

In comparison to the classical calculus let us mention that, for example, if $\mu \geq 0$, $t > 0$ and $\alpha > -1$, then the fractional derivative of the power function t^α is given by

$$D^\mu t^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \mu + 1)} t^{\alpha - \mu}. \tag{1}$$

Fig. 1 illustrates the 1/2-fractional derivative of $f(t) = t^2$. The three-dimensional graph of the fractional derivatives of $f(t) = t^2$ of order $0 \leq \mu \leq 2$ on the interval $[0, 2]$ is given on Fig. 2. These figures visualize the fact that the fractional derivatives are "continuously" distributed between the standard integer-order derivatives. Another illustration of the continuity of the fractional differential operator is provided on Fig. 3. It shows the changes of $D^\mu t^2$ for $0 \leq \mu \leq 2$ for a fixed $t \in [0, 2]$, i.e., the projection of the three-dimensional graph from Fig. 2 to the $(\mu, D^\mu t^2)$-plane.

The *Leibniz rule for fractional differentiation* is of primary importance for our considerations in this paper. It is well known [7], that if $f(\tau)$ is continuous in $[0, t]$ and $\varphi(\tau)$ has $n + 1$ continuous derivatives in $[0, t]$, then the fractional derivative of the product $\varphi(t)f(t)$ is given by the Leibniz rule for fractional differentiation

$$D^\mu [\varphi(t)f(t)] = \sum_{k=0}^{m} \binom{\mu}{k} \varphi^{(k)}(t) D^{\mu-k} f(t) - R_m^\mu(t),$$

where $\mu > 0$, $m \geq \mu + 1$ and

$$R_m^\mu(t) = \frac{1}{m! \Gamma(-\mu)} \int_0^t (t - \tau)^{-\mu-1} f(\tau) \int_\tau^t \varphi^{(m+1)}(\xi)(t - \xi)^n d\xi d\tau.$$