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On some fractional evolution equations

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ABSTRACT

In this paper the solutions of some evolution equations with fractional orders in a Banach space are considered. Conditions are given which ensure the existence of a resolvent operator for an evolution equation in a Banach space.

1. Introduction

In this paper we shall be concerned with a fractional evolution equation of the form

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} - A(t)x(t) = \int_0^t B(t,s)x(s)ds + f(t), \quad t > 0 \tag{1.1}$$

with the initial condition

$$x(0) = 0 \tag{1.2}$$

where $0 < \alpha \leq 1$, for each $t \in [0, T]$, $T > 0$, $A(t)$ is a closed linear operator with dense domain $D(A)$ which is independent of t. Also, for $0 \leq s \leq t \leq T$, $B(t, s)$ is a closed linear operator with domain at least $D(A)$. Suppose Y is the Banach space formed from $D(A)$ with the graph norm $\|y\|_Y = \|A(0)y\| + \|y\|$ where $\| \|$ is the norm on the Banach space X. As $A(t)$ and $B(t, s)$ are closed operators it follows that $A(t)$ and $B(t, s)$ are in the set of bounded operators from Y to X, $\beta(Y, X)$, for $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively. Assume further that $A(t)$ and $B(t, s)$ are continuous on $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively, into $\beta(Y, X)$. The function $f : R^+ \to X$ is absolutely continuous. Let $B(E)$ denote the Banach space of all linear bounded operators in E endowed with the topology defined by the operator norm. It is assumed that the operator $[A(t) + \lambda I]^{-1}$ exists in $B(E)$ for any λ with $Re\lambda \geq 0$ and $\|[A(t) + \lambda I]^{-1}\| \leq \frac{c}{|\lambda|+1}$ for each $t \in [0, T]$, where c is a positive constant independent both of t and of λ. Under these conditions each operator $A(s)$, $s \in [0, T]$, generates an analytic semigroup $e^{-tA(s)}$, $t > 0$, and there exists a positive constant C independent both of t and of s such that

$$\|A^n(s)e^{-tA(s)}\| \leq \frac{C}{t^n}$$

where $n = 0, 1$, $t > 0$, $s \in [0, T]$.

It is further assumed that $\{T(t)\}_{t \geq 0}$ defined by $T(t)f(s) = f(t + s)$ is a C_0 semigroup on χ with generator D_s on domain $D(D_s)$, where χ is a subspace of the set of bounded uniformly continuous functions on R^+ into X. Let us suppose that:

(H_0) $\{A(t)\}$, $0 \leq t \leq T$, is a stable family of generators such that $A(t)x$ is strongly continuously differentiable on $[0, T]$ for $x \in D(A)$. In addition, $B(t)x$ is strongly continuously differentiable on $[0, T]$ for $x \in D(A)$; see [1–6].

(H_1) $B(t)$ is continuous on $[0, \infty)$ into $\beta(Y, \chi)$.

(H_2) $B(t), B'(t) : Y \to D(D_s)$ for all $t, s \geq 0$.

(H_3) $D_sB(t), D_sB'(t)$ are continuous on $[0, \infty)$ into $\beta(Y, \chi)$.

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We shall first consider the fractional evolution equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} - A(t)x(t)\right) = B_1(t)x(t) + \int_0^t B_2(t,s)x(s)\mathrm{d}s + f_1(t) \tag{1.3}$$

with the initial conditions

$$x(0) = 0, \qquad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = 0 \tag{1.4}$$

and then we obtain the solution of the problem (1.1) and (1.2).

2. Existence of solutions

Theorem 2.1. *Assume that $B_1(t)$ and $B_2(t,s)$ are closed linear operators defined on dense sets in X into X with domain at least $D(A)$ and B_1 and B_2 satisfy* (H_0)–(H_3), *and $f_1 : R^+ \to X$ is absolutely continuous; then the problem (1.3) and (1.4) has a unique solution*

$$x(t) = \int_0^t \int_0^{\eta} \psi(t-\eta,\eta)R(\eta,\eta_1)f_1(\eta_1)\mathrm{d}\eta_1\mathrm{d}\eta + \int_0^t \int_0^{\eta}\int_0^s \psi(t-\eta,\eta)\phi(\eta,s)R(s,\eta_1)f_1(\eta_1)\mathrm{d}\eta_1\mathrm{d}s\mathrm{d}\eta.$$

Proof. First we assume that

$$\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} - A(t)x(t) = U(t). \tag{2.1}$$

Hence formally, from [7],

$$x(t) = \int_0^t \psi(t-\eta,\eta)U(\eta)\mathrm{d}\eta + \int_0^t \int_0^{\eta} \psi(t-\eta,\eta)\phi(\eta,s)U(s)\mathrm{d}s\mathrm{d}\eta \tag{2.2}$$

where $\psi(t,s) = \alpha\int_0^{\infty}\theta t^{\alpha-1}\zeta_{\alpha}(\theta)\mathrm{e}^{-t^{\alpha}\theta A(s)}\mathrm{d}\theta$, ζ_{α} is a probability density function defined on $[0,\infty]$, and ϕ is the unique solution of the integral equation

$$\phi(t,\tau) = \phi_1(t,\tau) + \int_{\tau}^t \phi(t,s)\phi_1(s,\tau)\mathrm{d}s,$$

$$\phi_1(t,\tau) = [A(t) - A(\tau)]\psi(t-\tau,\tau).$$

From (1.3), (1.4) and (2.2) we get

$$\begin{aligned}\frac{\mathrm{d}U(t)}{\mathrm{d}t} &= \int_0^t B_1(t)\psi(t-\eta,\eta)U(\eta)\mathrm{d}\eta + \int_0^t\int_0^{\eta} B_1(t)\psi(t-\eta,\eta)\phi(\eta,s)U(s)\mathrm{d}s\mathrm{d}\eta \\ &\quad + \int_0^t\int_0^z B_2(t,z)\psi(z-\eta,\eta)U(\eta)\mathrm{d}\eta\mathrm{d}z \\ &\quad + \int_0^t\int_0^z\int_0^{\eta} B_2(t,z)\psi(z-\eta,\eta)\phi(\eta,s)U(s)\mathrm{d}s\mathrm{d}\eta\mathrm{d}z + f_1(t) \end{aligned}\tag{2.3}$$

$$U(o) = 0. \tag{2.4}$$

We rewrite Eq. (2.3) as

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}t} &= \int_0^t C_1(t,\eta)U(\eta)\mathrm{d}\eta + \int_0^t C_2(t,s)U(s)\mathrm{d}s + \int_0^t C_3(t,\eta)U(\eta)\mathrm{d}\eta + \int_0^t C_4(t,s)U(s)\mathrm{d}s + f_1(t) \\ &= \int_0^t C_5(t,\eta)U(\eta)\mathrm{d}\eta + \int_0^t C_6(t,s)U(s)\mathrm{d}s + f_1(t) \\ &= \int_0^t C_7(t,s)U(s)\mathrm{d}s + f_1(t)\end{aligned}\tag{2.5}$$

where

$$\begin{aligned} C_1(t,\eta) &= B_1(t)\psi(t-\eta,\eta), C_2(t,s) = \int_t^s B_1(t)\psi(t-\eta,\eta)\phi(\eta,s)\mathrm{d}\eta, C_3(t,\eta) \\ &= \int_{\eta}^t B_2(t,z)\psi(z-\eta,\eta)\mathrm{d}z, C_4(t,s) = \int_s^t\int_z^s B_2(t,z)\psi(z-\eta,\eta)\phi(\eta,s)\mathrm{d}\eta\mathrm{d}z, C_5(t,\eta) \\ &= C_1(t,\eta) + C_3(t,\eta), C_6(t,s) = C_2(t,s) + C_4(t,s) \quad \text{and} \quad C_7(t,s) = C_5(t,s) + C_6(t,s).\end{aligned}\tag{2.6}$$