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# A fixed-point method for a class of super-large scale nonlinear complementarity problems

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ABSTRACT

We consider a class of complementarity problems involving functions which are nonlinear. In this paper we reformulate this nonlinear complementarity problem as a system of absolute value equations (which is nonsmooth). Then we propose a fixed-point method to solve this nonsmooth system. We prove that the proposed method is globally linearly convergent under a mild condition. The proposed method is greatly effective not only for small and medium size problems, but also for large and super-large scale problems. Especially, our method can efficiently solve super-large scale problems, with a million variables, in a few tens of minutes on a PC.



## 1. Introduction

Let $F : \Re^n \to \Re^n$ be defined by

$$F(x) = Mx + \Psi(x),$$

where $M = (m_{ij}) \in \Re^{n\times n}$ is an $M$-matrix, and $\Psi : \Re^n \to \Re^n$ is a monotone diagonal function. Recall that $M$ is called an $M$-matrix if it has nonpositive off-diagonals and $M^{-1}$ exists with $M^{-1} \geq 0$; and that a function $\Psi$ is called a monotonic diagonal function if the $i$th component of $\Psi$ is a function such that $\Psi_i(x) = \psi_i(x_i)$ where $\psi_i : \Re \to \Re$, and

$$(\psi_i(x_i) - \psi_i(y_i))(x_i - y_i) \geq 0, \quad i = 1, 2, \ldots, n.$$

In this paper, we consider a class of nonlinear complementarity problems, denoted by NCP(F), which is to find a vector $x \in \Re^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \text{and} \quad x^T F(x) = 0, \tag{1.1}$$

here, the inequalities are meant componentwise.

The NCP(F) (1.1) has many real world applications on some every important problems in physics and finance, such as the reaction–diffusion problems [1,2], the nonlinear parabolic complementarity problems [3], and European and American option valuation [4,5]. By using a discretization method such as the center difference, the piecewise linear finite element, or a finite volume method [3,5], we can reduce these problems into the NCP(F) (1.1) which can be modeled as free boundary value problems. Then a numerical solution of these problems is obtained by solving the NCP(F) (1.1). However, the resulting problem (1.1) is usually very large.

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**Example** ([2,6]). Let $\mathcal{D}$ be a bounded open set in $\Re^2$ with Lipschitz boundary $\partial\mathcal{D}$. Given two numbers $\lambda > 0$ and $p > 0$, consider the following free boundary value problem:

$$\begin{aligned} -\Delta u + \lambda u^p &= 0 \quad && \text{in } \mathcal{D}_+, \\ u &= 0 && \text{in } \mathcal{D}_0, \\ u = |\nabla u| &= 0 && \text{on } \Gamma, \\ u &= 1 && \text{on } \partial\mathcal{D}, \end{aligned}$$

where $\mathcal{D}_+ = \{z \in \mathcal{D} : u(z) > 0\}$, $\mathcal{D}_0 = \{z \in \mathcal{D} : u(z) = 0\}$, and $\Gamma = \partial\mathcal{D}_0 = \partial\mathcal{D}_+ \cap \mathcal{D}$ are unknown, and $\Delta u = \frac{\partial^2 u}{\partial z_1^2} + \frac{\partial^2 u}{\partial z_2^2}$. Using the five-point finite difference approximation, we can get a nonlinear complementarity problem with $F(x) = Mx + \Psi(x)$, where $M$ is an $M$-matrix and $\Psi(x)$ is a monotonically increasing diagonal function.

Over the last decade, a number of methods for solving complementarity problems have been developed. For instance, smoothing Newton methods [7–9] and semismooth Newton methods ([10,11] and references therein). In addition, the smoothing Newton method has been extensively studied in the literature including non-interior point continuation methods (refer to [12–21]). For related Newton methods for solving the NCP(F) (1.1) arising from free boundary value problems, please refer to [10,22,23]. Other methods about free boundary value problems can also be found in other papers (refer for instance to [2,6,24]).

Recently, Zhao and Wang [22] presented a full-Newton step non-interior point continuous algorithm to solve the NCP(F) (1.1) with Lipschitz functions. By using a smoothing approximation of plus function, the authors reformulated this complementarity problem as a system of equations and obtained a full-length Newton direction in each iteration based on some properties of the $M$-matrix. Generally, it needs the line search to obtain the iteration step-length for the non-interior point continuous algorithm. However, the authors used a full-Newton step in each iteration instead of a line search to obtain that the iteration sequence converges monotonically. Moreover, they showed that the algorithm is globally linearly and locally superlinearly convergent without any additional assumption, and locally quadratically convergent under suitable assumptions.

In Ref. [10], the authors proposed a modified semismooth Newton method to solve the NCP(F) (1.1) with Lipschitz functions by using some properties of the $M$-matrix. They reformulated the NCP(F) (1.1) as a system of nonsmooth equations and constructed a slant function of this system (for additional details, please refer to [10]) in order to find a Newton direction in each iteration based on this slant function. The author proved that the method is globally monotonically and locally superlinearly convergent.

G. Zhou, L. Caccetta and K.K. Teo [23] proposed a smoothing algorithm to solve the NCP(F) (1.1) with non-Lipschitz functions. They gave some equivalent formulations and reformulated the NCP(F) (1.1) as a nonlinear complementarity problem involving functions which are Lipschitz continuous by using the techniques in [24]. This is a crucial step in the development of the Newton-type method for a class of non-Lipschitz complementarity problems. In addition, the authors showed that the NCP(F) (1.1) has a unique solution under the condition that $M$ is a $P_0$-matrix. They proved that the method has global and local quadratic convergence properties.

The algorithms mentioned above have global and local fast convergence properties and are computationally efficient. It seems to be perfect not only in theoretical analysis, but also in numerical practice about these methods. However, we find that although these methods can readily handle small and medium size problems, they cannot solve large-scale problems (such as 300 thousand or 500 thousand dimensions) in short time, to say nothing of super-large scale problems (such as a million dimensions). In this paper, taking advantage of properties of the $M$-matrix, we propose a fixed-point algorithm that can deal with large and super-large scale problems in a short time. We prove that the proposed method is globally linearly convergent under mild assumptions. Especially, by a number of numerical experiments, we show that our method has wonderful effects on computational practice. It is good and efficient not only for small and medium size problems but also for large scale and super-large scale problems. Compared with the methods mentioned above for small and medium scale problems, our method is superior in solving these problems with high accuracy. It should be particularly pointed out that, for some discrete cases from free boundary value problems, our method can solve super-large scale NCP(F) (1.1) with $n$ on the order of a million in a few tens of minutes on a PC.

This paper is organized as follows. In the next section, we reformulate the NCP(F) (1.1) as an absolute value equation, and a fixed-point algorithm is applied to solve this equation. Then we discuss the convergence properties of the fixed-point algorithm for Lipschitz and non-Lipschitz complementarity problems in Sections 2.1 and 2.2, respectively. Preliminary numerical results are reported in Section 3.

In our notation, all vectors are column vectors, the subscript $T$ denotes transpose. $\Re^n$ denotes the space of $n$-dimensional real column vectors, and $\Re^n_+$ (respectively, $\Re^n_{++}$) denotes the nonnegative (respectively, positive) orthant in $\Re^n$. The symbol $\|\cdot\|_2$ (for convenience, we denote this symbol by $\|\cdot\|$) stands for the Euclidean norm in $\Re^n$. The matrix $I$ represents the identity matrix of appropriate dimension. For any vector $x \in \Re^n$, we denote by $x_i$ the $i$th component of $x$ and by $|x|$ the vector with absolute values of each component of $x$. We denote by $\mathrm{diag}\{x_1, \ldots, x_n\}$ (or $\mathrm{diag}\{x_i\}$) the diagonal matrix in $\Re^{n\times n}$ whose $i$th diagonal element is $x_i$.