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A meshless interpolation algorithm using a cell-based searching procedure

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ABSTRACT

In this paper we propose a fast algorithm for bivariate interpolation of large scattered data sets. It is based on the partition of unity method for constructing a global interpolant by blending radial basis functions as local approximants and using locally supported weight functions. The partition of unity algorithm is efficiently implemented and optimized by connecting the method with an effective cell-based searching procedure. More precisely, we construct a cell structure, which partitions the domain and strictly depends on the dimension of the subdomains, thus providing a meaningful improvement in the searching process compared to the nearest neighbour searching techniques presented in Allasia et al. (2011) and Cavoretto and De Rossi (2010, 2012). In fact, this efficient algorithm and, in particular, the new searching procedure enable us a fast computation also in several applications, where the amount of data to be interpolated is often very large, up to many thousands or even millions of points. Analysis of computational complexity shows the high efficiency of the proposed interpolation algorithm. This is also supported by numerical experiments.

1. Introduction

In the last decades, efficient methods and algorithms using radial basis functions (RBFs) have gained popularity in various areas of scientific computing such as multivariate interpolation, approximation theory, meshfree (or meshless) methods, neural networks, computer graphics, computer aided geometric design (CAGD) and machine learning. In particular, the need of having fast algorithms and powerful and flexible software is of great interest mainly in applications, where the amount of data to be interpolated is often very large, say many thousands or even millions of points (see, e.g., [1–5] for an overview).

In the literature, several techniques and alternative approaches have been proposed to have stable and accurate numerical algorithms (see, e.g., [6–10] and references therein), but, except for [11], none allows to deal with a truly great number of data in a relatively small amount of time.

In this paper we focus on the problem of constructing a new fast algorithm for bivariate interpolation of large sets of scattered data. It is based on the *partition of unity method* for constructing a global interpolant by blending radial basis functions as local approximants and using locally supported weight functions. The partition of unity method was firstly suggested in [12,13] in the mid 1990s in the context of meshfree Galerkin methods for the solution of partial differential equations (PDEs), but now it is also an effective method for fast computation in the field of approximation theory (see, for example, [2,14,5]). Similar local approaches involving the modified Shepard's method have already been studied in previous works (see, e.g., [15–21]).

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Thus, starting from the results of our previous researches (see [15,22–26]) where efficient searching procedures based either on the partition of a plane domain in strips or on the partition of a sphere in spherical zones are considered, we extend the previous ideas replacing the strip-based partition structure with a cell-based one. The latter leads to the creation of a *cell-based searching procedure*, whose origin comes from the repeated use of a *quicksort* routine with respect to different directions, enabling us to pass from unordered to ordered data structures. In particular, this process turns out to be strictly related to the construction of a partition of the domain Ω in square cells. It consists in generating two orthogonal families of parallel strips, namely a crossed-strip structure, where the original data set is suitably split up into ordered and well-organized data subsets.

Now, exploiting the ordered data structure and the domain partition, the crossed-strip algorithm is efficiently implemented and optimized by connecting the interpolation method with the effective cell-based searching procedure. More precisely, the technique is characterized by the construction of a double structure of crossed strips, called *cell structure*. It partitions the domain Ω in square cells and strictly depends on the dimension of its subdomains, providing a meaningful improvement in the searching procedures of the nearest neighbour points compared to the searching techniques presented in [15,22,24]. The final result is an efficient algorithm for bivariate interpolation of generally scattered data points, whose construction process can briefly be summarized in three stages: (i) partition the domain Ω into a suitable number of cells; (ii) consider an optimized cell-based searching procedure establishing the minimal number of cells to be examined, in order to localize the subset of nodes belonging to each subdomain; (iii) apply the partition of unity method combined with local radial basis functions.

Finally, an analysis of computational complexity shows the high efficiency of this interpolation algorithm, enabling a fast computation of a very large amount of data as shown by several numerical experiments.

The paper is organized as follows. In Section 2 we recall some theoretical preliminaries: at first, we discuss the solvability of the interpolation problem, referring to existence and uniqueness of radial basis function interpolants, we then give a general description of the partition of unity method, which uses radial basis functions as local approximants. In Section 3, we present in detail the cell-based partition algorithm for bivariate interpolation of generally scattered data points, which is efficiently implemented and optimized by using a nearest neighbour searching procedure. Computational complexity and storage requirements of the interpolation algorithm are analysed. In Section 4, we show numerical results concerning efficiency and accuracy of the partition of unity algorithm, while Section 5 contains an application to real data. Finally, Section 6 deals with conclusions and future work.

2. Preliminaries

2.1. Radial basis function interpolation

Let $\mathcal{X}_n = \{\mathbf{x}_i,\ i = 1, 2, \ldots, n\}$ be a set of distinct data points or nodes, arbitrarily distributed in a domain $\Omega \subseteq \mathbb{R}^N,\ N \geq 1$, with an associated set $\mathcal{F}_n = \{f_i,\ i = 1, 2, \ldots, n\}$ of data values or function values, which are obtained by sampling some (unknown) function $f : \Omega \to \mathbb{R}$ at the nodes, i.e., $f_i = f(\mathbf{x}_i),\ i = 1, 2, \ldots, n$. Thus, we can now give a precise formulation of the scattered data interpolation problem.

Problem 2.1. Given the point sets $\mathcal{X}_n$ and $\mathcal{F}_n$, find a (continuous) function $R : \Omega \to \mathbb{R}$ such that

$$R(\mathbf{x}_i) = f_i, \quad i = 1, 2, \ldots, n. \tag{1}$$

Now, using a RBF expansion to solve the scattered data interpolation problem in Ω, the above-mentioned problem can be written as follows.

Definition 2.1. Given the point sets $\mathcal{X}_n$ and $\mathcal{F}_n$, a radial basis function interpolant $R : \Omega \to \mathbb{R}$ assumes the form

$$R(\mathbf{x}) = \sum_{j=1}^{n} c_j \phi(d(\mathbf{x}, \mathbf{x}_j)), \quad \mathbf{x} \in \Omega, \tag{2}$$

where $d(\mathbf{x}, \mathbf{x}_j) = \|\mathbf{x} - \mathbf{x}_j\|_2$ is the Euclidean distance, $\phi : [0, \infty) \to \mathbb{R}$ is called *radial basis function*, and R satisfies the interpolation conditions (1).

Solving the interpolation problem under this assumption leads to a system of linear equations of the form

$$A\mathbf{c} = \mathbf{f},$$

where the entries of the interpolation matrix $A \in \mathbb{R}^{n \times n}$ are given by

$$A_{i,j} = \phi(d(\mathbf{x}_i, \mathbf{x}_j)), \quad i, j = 1, 2, \ldots, n, \tag{3}$$

$\mathbf{c} = [c_1, c_2, \ldots, c_n]^T$, and $\mathbf{f} = [f_1, f_2, \ldots, f_n]^T$. Then, the interpolation problem is well-posed, i.e., a solution to the problem exists and is unique in the interpolation space

$$T_\phi = \text{span}\{\phi(d(\cdot, \mathbf{x}_1)), \ldots, \phi(d(\cdot, \mathbf{x}_n))\}$$