



Simultaneous determination of time-dependent coefficients in the heat equation



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ABSTRACT

In this paper, the determination of time-dependent leading and lower-order thermal coefficients is investigated. We consider the inverse and ill-posed nonlinear problems of simultaneous identification of a couple of these coefficients in the one-dimensional heat equation from Cauchy boundary data. Unique solvability theorems of these inverse problems are supplied and, in one new case where they were not previously provided, are rigorously proved. However, since the problems are still ill-posed the solution needs to be regularized. Therefore, in order to obtain a stable solution, a regularized nonlinear least-squares objective function is minimized in order to retrieve the unknown coefficients. The stability of numerical results is investigated for several test examples with respect to different noise levels and for various regularization parameters. This study will be significant to researchers working on computational and mathematical methods for solving inverse coefficient identification problems with applications in heat transfer and porous media.

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1. Introduction

Simultaneous determination of several unknown coefficients in parabolic partial differential equations has been investigated in various studies in the past, see e.g. the monographs of Prilepko et al. [1] and Ivanchov [2]. In heat conduction for example, attention was paid to the unique solvability of one-dimensional inverse problems for the heat equation in the case when the unknown thermal coefficients are constant [3], time-dependent [4,5], space-dependent [6], or temperature-dependent [7–9]. In these papers, the authors investigated the existence and uniqueness of the solution of the inverse problem, though no numerical method/solution was presented.

When solving an inverse problem the choice of additional information about the solution is crucial since this information enables us to determine the unknown parameters of the process under consideration uniquely. Usually, this additional information/observation is given by the boundary conditions or, the value of the solution on a specific subdomain or, at a certain time [10]. In [11], the authors proposed a new algorithm based on space decomposition in a reproducing kernel space for solving the inverse problem of finding the time-dependent thermal diffusivity. In [12,13] the problem of finding the time-dependent leading coefficient and temperature distribution with Dirichlet boundary conditions and measured heat flux as the overdetermination condition was considered. In [14], the author considered retrieving lower-order time-dependent coefficients using the Trace Type Functional approach [15], which assumes that the governing partial differential equation is valid at the boundary. However, this approach does not seem so stable [16], and it has never been applied to inverse coefficient identification problems in which the unknown coefficients appear at the leading order in the heat operator.

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In this paper, we investigate the inverse problems of simultaneous determination of time-dependent leading and lower-order thermal coefficients. The paper is organized as follows. In the next section, we give the mathematical formulations of three inverse problems for which the unique solvability theorems of [12,4] are stated and, in one case, proved. The numerical finite-difference discretization of the direct problem is described in Section 3, whilst Section 4 introduces the regularized nonlinear minimization used for solving in a stable manner the inverse problems under investigation. In Section 5, we provide numerical results and discussion. Finally, conclusions are presented in Section 6.

2. Mathematical formulations of the inverse problems

Consider the linear one-dimensional parabolic equation with time-dependent coefficients

$$C(t) \frac{\partial u}{\partial t}(x, t) = K(t) \frac{\partial^2 u}{\partial x^2}(x, t) + Q(t) \frac{\partial u}{\partial x}(x, t), \quad (x, t) \in (0, \ell) \times (0, T) =: \Omega \quad (1)$$

where, in heat conduction, u represents the temperature in a finite slab of length $\ell > 0$ recorded over the time interval $(0, T)$ with $T > 0$, C and K represent the heat capacity and thermal conductivity of the heat conductor, respectively, $Q(t) = c(t)v(t)$ with c and v representing the heat capacity and velocity of a fluid flowing through the heat conducting body [17,18]. The first term on the right-hand side of Eq. (1) represents the diffusion, whilst the second term, if $v(t)$ is positive, represents the convection. A similar situation occurs in porous media, [19], where the properties are referred to as hydraulic rather than thermal as in heat transfer. For example, in the contaminant transport in groundwater the first term on the right-hand side of Eq. (1) represents the dispersion of the contaminant as it moves through the porous medium, whilst the second term with $v(t)$ negative describes the advection of the contaminant which flows along with the bulk movement of groundwater.

The initial condition is

$$u(x, 0) = \phi(x), \quad x \in [0, \ell], \quad (2)$$

and the boundary and over-determination conditions are

$$u(0, t) = \mu_1(t), \quad u(\ell, t) = \mu_2(t), \quad t \in [0, T], \quad (3)$$

$$-K(t)u_x(0, t) = v_1(t), \quad K(t)u_x(\ell, t) = v_2(t), \quad t \in [0, T]. \quad (4)$$

Conditions (3) and (4) represent the specification of the boundary temperature and heat flux, respectively. Together they represent the Cauchy data for the inverse coefficient identification problems (ICIPs) which are described next.

We distinguish three ICIPs covering the simultaneous determination of a couple of coefficients in (1). The case of identifying all three coefficients in (1) is deferred to a future work.

2.1. Inverse problem 1

Assuming that $c(t)v(t) = 0$, the inverse problem 1 (IP1) requires the simultaneous determination of the time-dependent thermal conductivity $K(t) > 0$, the heat capacity $C(t) > 0$ and the temperature $u(x, t)$ satisfying the one-dimensional heat equation

$$C(t) \frac{\partial u}{\partial t}(x, t) = K(t) \frac{\partial^2 u}{\partial x^2}(x, t), \quad (x, t) \in \Omega \quad (5)$$

subject to the initial and boundary conditions (2)–(4).

For this IP1 we have the following existence and uniqueness of solution theorems [4].

Theorem 1 (Existence). Suppose that:

1. $\phi \in C^2[0, \ell]$ and $\mu_i, v_i \in C^1[0, T]$ for $i = 1, 2$.
2. The consistency conditions are satisfied:

$$\mu_1(0) = \phi(0), \quad \mu_2(0) = \phi(\ell), \quad -v_1(0)\phi'(\ell) = v_2(0)\phi'(0), \quad \mu_1'(0)\phi''(\ell) = \mu_2'(0)\phi''(0).$$

3. The following conditions are satisfied:

$$\begin{aligned} \phi'(x) &\geq 0, \quad x \in [0, \ell], \quad \phi''(x) + \phi''(\ell - x) > 0, \quad x \in [0, \ell/2], \\ v_1^2(t) + v_2^2(t) &> 0, \quad \mu_2'(t) - \mu_1'(t) \geq 0, \quad (1 + \chi(t))\mu_1'(t) + (1 - \chi(t))\mu_2'(t) > 0, \\ \chi(t) &> 0, \quad \chi'(t) \geq 0, \quad t \in [0, T], \\ (1 + \chi(t))\phi''(x) + (1 - \chi(t))\phi''(\ell - x) &> 0, \quad x \in [0, \ell/2], \quad t \in [0, T], \\ \phi''(x) - \phi''(\ell - x) &\geq 0, \quad \text{or} \quad \phi''(x) - \phi''(\ell - x) \leq 0, \quad x \in [0, \ell/2], \end{aligned}$$

where $\chi(t) = \frac{v_2(t) + v_1(t)}{v_1(t) - v_2(t)}$. Then, for a sufficiently small $T > 0$, the inverse problem (2)–(5) has at least one solution $\{C(t), K(t), u(x, t)\}$, where the functions $C(t)$ and $K(t)$ are continuous and positive on $[0, T]$ and $u(x, t)$ belongs to the class $C^{2,1}(\Omega) \cap C^{1,0}(\bar{\Omega})$.

Theorem 2 (Uniqueness). Suppose that the following conditions are satisfied:

1. $\phi \in C^2[0, \ell]$, $\mu_i \in C^1[0, T]$ and $v_i \in C[0, T]$ for $i = 1, 2$;

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