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An octahedral equal area partition of the sphere and near optimal configurations of points

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ABSTRACT

We construct a new area preserving map from the unit sphere to the regular octahedron, both centered at the origin. Its inverse map allows the construction of uniform and refinable grids on a sphere, starting from any triangular uniform and refinable grid on the faces of the octahedron. We prove that our new grids are diameter bounded and then, for the resulting configurations of points we calculate some Riesz s-energies and we compare them with the optimal ones. For some configurations, we also calculate the point energies and we list the minimum and maximum values, concluding that these values are very close. Finally, we show how we can map a hemisphere of the Earth onto a square, using our new area preserving projection. The simplicity and the symmetry of our formulas lead to fast computations.

1. Introduction

A uniform grid on a two-dimensional domain D is a grid all of whose cells have the same area. This fact is required in statistical applications and in construction of wavelet bases of the space $L^2(D)$, when we wish to use the standard 2-norm and inner product instead of a weighted norm dependent on the grid. A refinement process is needed for a multiresolution analysis or for multigrid methods, when a grid is not fine enough to solve a problem accurately. A uniform refinement consists in dividing each cell into a given number of smaller cells with the same area. To be efficient in practice, a refinement procedure should also be a simple one. In many applications, especially in geosciences, one requires simple, uniform and refinable (hierarchical) grids on the sphere. One simple method to construct such grids is to transfer existing planar grids.

The existence of partitions of $\mathbb{S}^2$ into regions of equal area and small diameter has been already used by Alexander [1], who derives lower bounds for the maximum sum of distances between points on the sphere. Based on the construction by Zhou [2], Leopardi derives a recursive zonal equal area sphere partitioning algorithm for the unit sphere $\mathbb{S}^d$ embedded in $\mathbb{R}^{d+1}$, see [3]. The constructed partition in [3] for $\mathbb{S}^2$ consists of polar cups and rectilinear regions that are arranged in zonal collars. Besides the problem that we have to deal with different kinds of regions, the obtained partition is not suitable for various applications where one needs to avoid that vertices of spherical rectangles lie on edges of neighboring rectangles.

Other constructions of equal area partitions of $\mathbb{S}^2$, used in astronomy, include the HEALPix grid [4], providing a hierarchical equal area iso-latitude pixelization, the truncated icosahedron-method of Snyder [5], the small circle subdivision method introduced in [6], the icosahedron-based method by Tegmark [7], see also [8]. For a larger list of such spherical grids, together with their properties, see Section 1 in [9]. A complete description of all known spherical projections from a sphere or parts of a sphere to the plane, used in cartography, is realized in [10,11]. We should mention that most of the existing constructions of spherical hierarchical grids do not provide an equal area partition. However, in [12,9,13] we have already constructed some area preserving maps to the sphere, which use the Lambert azimuthal equal area projections, and

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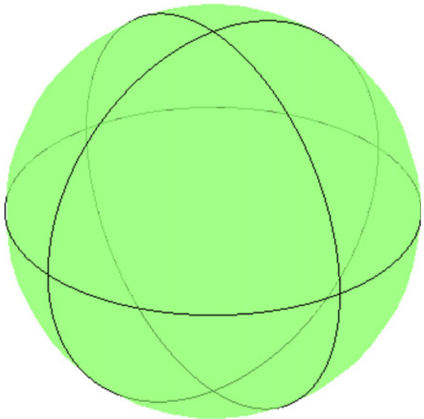

Fig. 1. The eight spherical triangles obtained as intersections of the coordinate planes with the sphere $\mathbb{S}^2$.

which allow the construction of uniform and refinable spherical grids. In this paper we propose a new area preserving map, without making use of Lambert's projection.

The paper is structured as follows: After introducing the notations in Section 2, in Section 3 we construct a new area preserving map from the unit sphere $\mathbb{S}^2$ to the regular octahedron, different from the one in [13] and using a different idea from the one in [9,13]. Formulas for the inverse are given in Section 4. Thus, *any* grid on the octahedron can be transported to the sphere. In Section 5 we prove that uniform subdivisions of the octahedron lead to diameter bounded partitions of the sphere. Then, by taking the vertices of the triangular cells we obtain configurations of $M = 4n^2 + 2$ points on the sphere and by taking their centers we obtain other configurations of $M = 8n^2$ points. Their union is also a configuration, of $12n^2 + 2$ points. For all these configurations, we calculate some Riesz s-energies and we compare them to some optimal values given in [14,15]. We also calculate the point energies and we list the minimum and maximum values. Since we give explicit formulas both for the map from the octahedron to the sphere, and from the sphere to the octahedron, the method is easy to implement. Also, the symmetry of our formulas leads to fast computations. For all these reasons, we believe that our construction may achieve an essential impact for different applications in geosciences.

2. Preliminaries

Consider the unit sphere $\mathbb{S}^2$ centered at the origin O and the regular octahedron $\mathbb{K}$ of the same area, centered at O and with vertices on the coordinate axes. Thus, the area of each face is $\pi/2$, the edge of the octahedron has the length

$$L = \frac{\sqrt{2\pi}}{\sqrt[4]{3}}, \tag{1}$$

and the distance from the origin to each vertex of $\mathbb{K}$ will therefore be $L/\sqrt{2}$.

Consider the parametric equations of the sphere

$$\begin{aligned} x &= \cos\theta \sin\varphi, \\ y &= \sin\theta \sin\varphi, \\ z &= \cos\varphi, \end{aligned} \tag{2}$$

where $\varphi \in [0, \pi]$ is the colatitude and $\theta \in [0, 2\pi)$ is the longitude. A simple calculation shows that the area element of the sphere is

$$dS = \sin\varphi \, d\theta \, d\varphi. \tag{3}$$

We cut the sphere with the coordinate planes $x = 0, y = 0, z = 0$ and obtain the spherical triangles in Fig. 1. Each face F_i of $\mathbb{K}$ is thus situated in one of the following domains:

$$\begin{aligned}
I_1 &= \{(x, y, z),\ x \ge 0,\ y \ge 0,\ z \ge 0\},\\
I_2 &= \{(x, y, z),\ x \ge 0,\ y \ge 0,\ z \le 0\},\\
I_3 &= \{(x, y, z),\ x \ge 0,\ y \le 0,\ z \ge 0\},\\
I_4 &= \{(x, y, z),\ x \ge 0,\ y \le 0,\ z \le 0\},\\
I_5 &= \{(x, y, z),\ x \le 0,\ y \ge 0,\ z \ge 0\},\\
I_6 &= \{(x, y, z),\ x \le 0,\ y \ge 0,\ z \le 0\},\\
I_7 &= \{(x, y, z),\ x \le 0,\ y \le 0,\ z \ge 0\},\\
I_8 &= \{(x, y, z),\ x \le 0,\ y \le 0,\ z \le 0\}.
\end{aligned}$$

More precisely, $F_i \subset I_i$ for $i = 1, \ldots, 8$. The portion of the sphere situated in I_i will be denoted by $\mathcal{F}_i$.