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Nonlinear oscillator with discontinuity by generalized harmonic balance method

A. Beléndez *, E. Gimeno, M.L. Alvarez, D.I. Méndez

Departamento de Física, Ingeniería de Sistemas y Teoría de la Señal, Universidad de Alicante, Apartado 99, E-03080 Alicante, Spain

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ABSTRACT

A generalized harmonic balance method is used to calculate the periodic solutions of a nonlinear oscillator with discontinuities for which the elastic force term is proportional to sgn(x). This method is a modification of the generalized harmonic balance method in which analytical approximate solutions have rational form. This approach gives us not only a truly periodic solution but also the frequency of the motion as a function of the amplitude of oscillation. We find that this method works very well for the whole range of amplitude of oscillation in the case of the antisymmetric, piecewise constant force oscillator and excellent agreement of the approximate frequencies with the exact one has been demonstrated and discussed. For the second-order approximation we have shown that the relative error in the analytical approximate frequency is 0.24%. We also compared the Fourier series expansions of the analytical approximate solution and the exact one. Comparison of the result obtained using this method with the exact ones reveals that this modified method is very effective and convenient.

1. Introduction

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion [1–4]. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem. There are several techniques used to find approximate solutions to nonlinear problems. Some of these techniques include perturbation [4–7], variational [8–13], decomposition [14], homotopy perturbation [15–29], homotopy analysis [30,31], harmonic balance [4], standard and modified Lindstedt–Poincaré [4,32–38], artificial parameter [37,38], linearized and quasilinearized harmonic balance [39–44] methods, etc. An excellent review on some asymptotic methods for strongly nonlinear equations can be found in detail in References [2] and [37].

In the present paper we obtain higher-order analytical approximations to the periodic solutions to a nonlinear oscillator with discontinuity for which the elastic restoring force is an antisymmetric and constant force. To do this, we apply a modified generalized harmonic balance method [4,45]. This type of oscillator has been analyzed by Özis and Yildirim [16] applying the first-order homotopy perturbation method and by Beléndez et al. [46] applying the higher-order homotopy perturbation method. This oscillator has been also studied by Rafei et al. [10] applying He's variational iteration method, by Liu [36] applying a modified Lindstedt–Poincaré method, by Wu et al. [42] using a linearized harmonic balance technique and by Ramos [47] using an artificial parameter Lindstedt–Poincaré method. Now we apply a modified generalized, rational harmonic balance method to obtain analytic approximate solutions for this nonlinear oscillator. The harmonic balance

* Corresponding author. Tel.: +34 96 5903651; fax: +34 96 5909750.
E-mail address: a.belendez@ua.es (A. Beléndez).

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method is a well-established method for the analysis of nonlinear problems, the time domain response of which can be expressed as a Fourier series. In the usual harmonic balance methods, the solution of a nonlinear system is assumed to be of the form of a truncated Fourier series [4]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. Being different from the other nonlinear analytical methods, such as perturbation techniques, the harmonic balance method does not depend on small parameters, so that it can find wide application in nonlinear problems without linearization or small perturbations. In the generalized or rational harmonic balance method, the approximate solution obtained approximates all of the harmonics in the exact solution [48], whereas the usual harmonic balance techniques provide an approximation to only the lowest harmonic components. In an attempt to provide better solution methodology a modification in this technique is proposed. As we will see, the second-order approximation obtained by this method is of extreme accuracy.

2. Solution procedure

Non-smooth oscillators play an important role in nonlinear dynamics [18,47,49–51]. Conservative non-smooth oscillators such as the one considered here are governed by

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{1}$$

where x is the displacement and $f(x)$ is a nonlinear, non-smooth function of x. We consider the case corresponding to the antisymmetric, piecewise constant force oscillator, $f(x) = \text{sgn}(x)$, which has been considered by several authors [4,10,16,36,42,46,47,50,51]

$$\frac{d^2x}{dt^2} + \text{sgn}(x) = 0 \tag{2}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \tag{3}$$

and sgn(x) is defined as

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ +1, & x > 0. \end{cases} \tag{4}$$

Eq. (2) models the motion of a punctual ball rolling in a "V" shape trough in a constant gravitational field. The arms of the "V" make equal angles with horizontal plane and the origin of the (horizontal) x coordinate is taken to be the point of interaction of the two arms [4]. In a suitable set of units, the equation of motion can be written as Eq. (2). All the solutions to Eq. (2) are periodic. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$, i.e., the functional behaviour of ω as a function of the initial amplitude A.

A new independent variable $\tau = \omega t$ is introduced. Then Eqs. (2) and (3) can be rewritten as

$$\omega^2 \frac{d^2x(\tau)}{d\tau^2} + \text{sgn}(x(\tau)) = 0, \quad x(0) = A, \qquad \frac{dx}{d\tau}(0) = 0. \tag{5}$$

The new independent variable is chosen in such a way that the solution of Eq. (5) is a periodic function of τ of period 2π.

Following the lowest-order harmonic balance approximation, we set

$$x_1(\tau) = A\cos\tau \tag{6}$$

which satisfies the initial conditions in Eq. (5). Substituting Eq. (6) into Eq. (5) and setting the resulting coefficient of $\cos\tau$ to zero yield the first approximation to the frequency in terms of A

$$\omega_1(A) = \frac{2}{\sqrt{\pi A}} \approx \frac{1.128379}{\sqrt{A}}, \qquad T_1(A) = \frac{2\pi}{\omega_1(A)} = 5.568328\sqrt{A}. \tag{7}$$

Eq. (7) is identical to the result that can be obtained from the application of a modified Lindstedt–Poincaré method [36], the homotopy perturbation method [16,46] and a variational iteration method [10,12].

In order to determine an improved approximation we use a generalized, rational form given by the following expression [4,45]

$$x_2(\tau) = \frac{A_1\cos\tau}{1 + B_2\cos 2\tau}. \tag{8}$$