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The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics

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ABSTRACT

Variational iteration method has been used to handle linear and nonlinear differential equations. The main property of the method lies in its flexibility and ability to solve nonlinear equations accurately and conveniently. In this work, a general framework of the variational iteration method is presented for analytical treatment of fractional partial differential equations in fluid mechanics. The fractional derivatives are described in the Caputo sense. Numerical illustrations that include the fractional wave equation, fractional Burgers equation, fractional KdV equation, fractional Klein–Gordon equation and fractional Boussinesq-like equation are investigated to show the pertinent features of the technique. Comparison of the results obtained by the variational iteration method with those obtained by Adomian decomposition method reveals that the first method is very effective and convenient. The basic idea described in this paper is expected to be further employed to solve other similar linear and nonlinear problems in fractional calculus.

1. Introduction

Recent advances of fractional differential equations are stimulated by new examples of applications in fluid mechanics, viscoelasticity, mathematical biology, electrochemistry and physics. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [1], and the fluid-dynamic traffic model with fractional derivatives [2] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in Ref. [3], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [4]. Different fractional partial differential equations have been studied and solved including the space–time fractional diffusion-wave equation [5–7], the fractional advection-dispersion equation [8,9], the fractional telegraph equation [10], the fractional KdV equation [11] and the linear inhomogeneous fractional partial differential equations [12].

The Adomian decomposition method [13–17] and the variational iteration method [18–38] are relatively new approaches to provide an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytical solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization. The decomposition method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations [13,14]. Recently, the application of the method is extended for fractional differential equations [10,11,39–44]. The variational iteration method, which proposed by Ji-Huan He [19–28], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [3] was the first to apply the

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variational iteration method to fractional differential equations. Recently Odibat and Momani [41–47] implemented the variational iteration method to solve linear and nonlinear differential equations of fractional order.

The objective of this paper is to extend the application of the variational iteration method to obtain analytical solutions to some fractional partial differential equations in fluid mechanics. These equations include wave equation, Burgers equation, KdV equation, Klein–Gordon equation and Boussinesq-like equation. The variational iteration method is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. The method introduces the solution in the form of a convergent fractional series with elegantly computable terms. The corresponding solutions of the integer order equations are found to follow as special cases of those of fractional order equations.

Throughout this paper, fractional partial differential equations are obtained from the corresponding integer order equations by replacing the first-order or the second-order time derivative by a fractional in the Caputo sense [48] of order α with $0 < \alpha \leq 1$ or $1 < \alpha \leq 2$.

2. Preliminaries and notations

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad \alpha > 0, t > 0,$$
$$J^0 f(t) = f(t).$$

Properties of the operator J^α can be found in [49–51], we mention only the following: For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [48].

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(t) \mathrm{d}t, \tag{2.1}$$

for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If $m - 1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D^\alpha J^\alpha f(t) = f(t),$$

and

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem [52]. In this paper, we consider the one-dimensional linear inhomogeneous fractional partial differential equations in fluid mechanics, where the unknown function $u(x, t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. The fractional derivative is taken in Caputo sense as follows: