Computers and Mathematics with Applications 58 (2009) 2236–2259

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Approximate solution of a mixed nonlinear stochastic oscillator

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ARTICLE INFO

Keywords:
Nonlinear stochastic differential equations
Wiener–Hermite expansion
WHEP technique
Homotopy perturbation
Mathematica

ABSTRACT

In this paper, nonlinear oscillators under mixed quadratic and cubic nonlinearities with stochastic inputs are considered. Different methods are used to obtain second order approximations, namely; the Wiener–Hermite and perturbation (WHEP) technique and the homotopy perturbation method (HPM). Some statistical moments are computed for the different methods using mathematica 5. Comparisons are illustrated through figures for different case-studies.

1. Introduction

Quadrate and cubic oscillations arise through many applied models in applied sciences and engineering when studying oscillatory systems [1]. These systems can be exposed to a lot of uncertainties through the external forces, the damping coefficient, the frequency and/or the initial or boundary conditions. These input uncertainties cause the output solution process to be also uncertain. For most of the cases, getting the probability density function (p.d.f.) of the solution process may be impossible. So, developing approximate techniques through which approximate statistical moments can be obtained, is an important and necessary work. There are many techniques which can be used to obtain statistical moments of such problems. The main goal of this paper is to introduce an approximate solution for the general mixed quadrate and cubic nonlinearities of an oscillatory problem. Section 2 deals with quadrate oscillations using two techniques, mainly; the WHEP technique and HPM. The cubic nonlinearity is analyzed in Section 3 using the same previous techniques. The general problem of mixed nonlinearities is solved approximately using only the homotopy perturbation method in Section 4. Some illustrations and comparisons are made to testify the method of analysis.

2. Quadrate nonlinearity

In this section, the following quadratic nonlinear oscillatory equation is considered:

$$\ddot{x}(t;\omega) + 2w\zeta\dot{x} + w^2x + \varepsilon w^2x^2 = F(t;\omega), \quad t \in [0, T] \tag{1}$$

under stochastic excitation $F(t;\omega)$ with deterministic initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0,$$

where w: frequency of oscillation,

ζ: damping coefficient

ε: deterministic nonlinearity scale

$\omega \in (\Omega, \sigma, P)$: a triple probability space with Ω as the sample space, σ is a σ-algebra on event in Ω and P is a probability measure.

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doi:10.1016/j.camwa.2009.03.057

Lemma 1. *The solution of Eq. (1), if exists, is a power series of* ε.

Proof. Rewriting Eq. (1), it can take the following form

$$\ddot{x}(t;\omega) + 2w\zeta\dot{x} + w^2 x = F(t) - \varepsilon w^2 x^2.$$

Following Pickard approximation, the equation can be rewritten as

$$\ddot{x}_{n+1}(t) + 2w\zeta\dot{x}_{n+1} + w^2 x_{n+1} = F(t) - \varepsilon w^2 x_n^2, \quad n \geq 0$$

where the solution at $n = 0$, x_0, is corresponding for the simple linear case at $\varepsilon = 0$.

At $n = 1$, the iteration takes the form:

$$\ddot{x}_1(t) + 2w\zeta\dot{x}_1 + w^2 x_1 = F(t) - \varepsilon w^2 x_0^2,$$

which has the following general solution

$$x_1(t) = \psi(t) - \varepsilon w^2 \int_0^t h(t-s) x_0^2(s) \mathrm{d}s,$$

or

$$x_1(t) = x_1^{(0)} + \varepsilon x_1^{(1)}.$$

At $n = 2$, the iteration takes the form:

$$\ddot{x}_2(t) + 2w\zeta\dot{x}_2 + w^2 x_2 = F(t) - \varepsilon w^2 x_1^2,$$

which has the following general solution

$$x_2(t) = x_2^{(0)} + \varepsilon x_2^{(1)} + \varepsilon^2 x_2^{(2)} + \varepsilon^3 x_2^{(3)}.$$

Proceeding like this, one can get the following

$$x_n(t) = x_n^{(0)} + \varepsilon x_n^{(1)} + \varepsilon^2 x_n^{(2)} + \varepsilon^3 x_n^{(3)} + \cdots + \varepsilon^{n+m} x_n^{(n+m)}.$$

Assuming the solution exists, it will be

$$x(t) = \lim_{n\to\infty} x_n(t) = \sum_{j=0}^{\infty} \varepsilon^j x_j,$$

which is a power series of ε. □

As a direct result of this lemma, it is expected that the average, the variance as well as the covariance are also power series of ε.

2.1. Applying WHEP technique

Since Meecham and his co-workers [2] developed a theory of turbulence involving a truncated Wiener–Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [3–8]. A lot of general applications in fluid mechanics was also studied in [9–11]. Scattering problems attracted the WHE applications through many authors [12–16]. The nonlinear oscillators were considered as an opened area for the applications of WHE as can be found in [17–23]. There are a lot of applications in boundary value problems [24,25] and generally in different mathematical studies [26–29].

The application of the WHE aims at finding a truncated series solution to the solution process of differential equations. The truncated series composes of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, always there exist difficulties in solving the resultant set of deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [22] using the perturbation technique to solve perturbed nonlinear problems.

The WHE method utilizes the Wiener–Hermite polynomials which are the elements of a complete set of statistically orthogonal random functions [30]. The Wiener–Hermite polynomial $H^{(i)}(t_1, t_2, \ldots, t_i)$ satisfies the following recurrence relation:

$$H^{(i)}(t_1, t_2, \ldots t_i) = H^{(i-1)}(t_1, t_2, \ldots t_{i-1}) \cdot H^{(1)}(t_i) - \sum_{m=1}^{i-1} H^{(i-2)}(t_1, t_2, \ldots t_{i-2}) \cdot \delta(t_{i-m} - t_i), \quad i \geq 2 \tag{2}$$