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A variational approach to nonlinear two-point boundary value problems

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ABSTRACT

A variational formulation is established for a nonlinear two-point boundary value problem, an analytical solution is obtained using the Ritz method, and the obtained solution is valid for the whole solution domain.

1. Introduction

Two-point boundary value problems arise in applied mathematics, theoretical physics, engineering, control and optimization theory etc. In this paper, we consider two-point boundary value problems of the following type [1].

$$y'' = f(x, y, y'), \quad a \le x \le b \quad y(a) = 0, \quad y(b) = 0. \tag{1}$$

The two-point boundary value problems were studied by Wazwaz using the decomposition method [2]. The main demerit of the decomposition method is the difficulty in calculating the so-called Adomian polynomials [3]. In this paper, a variational approach [4–13] is applied to the discussed problem.

2. Implementation of the variational approach

In order to use the variational method [4–13], we have to establish a variational formulation for the discussed problem first. To illustrate the solution procedure, we consider the following examples.

Example 1. We first consider the following boundary value problem

$$Y'' + \frac{2}{x}Y' + Y^3 + 3xY^2 + 3Yx^2 + x^3 + \frac{2}{x} - 6 - x^6 = 0 \tag{2}$$

with the boundary conditions

$$Y(0) = Y(1) = 0. \tag{3}$$

The above equation, by a simple transformation, $y = Y + x$, turns out to be Example 2 in Ref. [2] where the decomposition method was used.

Applying the semi-inverse method [11–13], we obtain the following variational functional

$$J[Y(x)] = \int_0^1 \left[-\frac{1}{2}x^2(Y')^2 + x^2\left(\frac{1}{4}Y^4 + xY^3 + \frac{3}{2}x^2Y^2 + \left(\frac{2}{x} - 6 + x^3 - x^6\right)Y\right)\right] dx. \tag{4}$$

Assume that the solution can be expressed in the following form

$$Y(x) = x(x-1)(ax+b) \tag{5}$$

where a and b represent unknown constants to be further determined.

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Submitting Eq. (5) into Eq. (4) and making $J[Y(x)]$ stationary with respect to a and b result in

$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{371}{3960} + \frac{1}{15015}a^3 + \frac{1}{2145}ab^2 + \frac{3}{10010}a^2b - \frac{92}{1155}a \\ &\quad + \frac{1}{3960}b^3 - \frac{11}{120}b - \frac{3}{2860}a^2 - \frac{1}{330}ab - \frac{1}{440}b^2 = 0 \end{aligned} \tag{6}$$

$$\frac{\partial J}{\partial b} = \frac{1151}{9240} + \frac{1}{2145}a^2b + \frac{1}{10010}a^3 + \frac{1}{1320}ab^2 + \frac{1}{2310}b^3 - \frac{17}{140}b - \frac{11}{120}a - \frac{1}{280}b^2 - \frac{1}{660}a^2 - \frac{1}{220}ab = 0. \tag{7}$$

Solving Eqs. (6) and (7) simultaneously, we can determine the values of a and b: $a = 0$ and $b = 1$.

We, therefore, obtain

$$Y(x) = x^2 - x \tag{8}$$

which is the exact solution.

The solution process is simpler than that of the decomposition method [2].

Example 2. We consider the following equation

$$Y'' + \frac{4}{x}(Y' + 1) + (Y + x + 2)^2 - 4 - 18x - 4x^3 - x^6 = 0 \tag{9}$$

with the boundary conditions

$$Y(0) = Y(1) = 0. \tag{10}$$

Eq. (9) becomes Example 3 in Ref. [2] by a transformation, $y = Y + x + 2$.

The variational formulation can be easily established using the semi-inverse method [11–13]

$$J[Y(x)] = \int_0^1 \left[-\frac{1}{2}x^4(Y')^2 + x^4\left(\frac{1}{3}(Y + x + 2)^3 + \left(\frac{4}{x} - 4 - 18x - 4x^3 - x^6\right)Y\right)\right] \mathrm{d}x. \tag{11}$$

Assume that the solution can be expressed as

$$Y(x) = x(x - 1)(ax + b) \tag{12}$$

where a and b are unknown constants to be further determined.

Proceeding in a similar way as before, we have

$$\frac{\partial J}{\partial a} = \frac{223}{1716} - \frac{19}{315}a - \frac{3}{44}b - \frac{1}{4004}a^2 - \frac{1}{1430}ab - \frac{1}{1980}b^2 = 0 \tag{13}$$

$$\frac{\partial J}{\partial b} = \frac{719}{4680} - \frac{1}{12}b - \frac{3}{44}a - \frac{1}{1320}b^2 - \frac{1}{2860}a^2 - \frac{1}{990}ab = 0. \tag{14}$$

Solving Eqs. (13) and (14) simultaneously yields a and b.

We, therefore, obtain

$$Y(x) = x^3 - x \tag{15}$$

which is the exact solution.

Example 3. As the last example, we consider the following boundary value problem [2]

$$y'' + \pi^3\frac{y^2}{\sin(\pi x)} = 0 \tag{16}$$

with the boundary conditions

$$y(0) = y(1) = 0. \tag{17}$$

Its variational formulation reads

$$J[Y(x)] = \int_0^1 \left[-\frac{1}{2}(Y')^2 + \frac{\pi^3 y^3}{3\sin(\pi x)}\right] \mathrm{d}x. \tag{18}$$

Assume that the solution has the following form

$$Y(x) = x(x - 1)(ax + b) \tag{19}$$

where a and b are unknown constants to be further determined.