Computers and Mathematics with Applications 58 (2009) 2489–2494

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# A new application of He's variational iteration method for the solution of the one-phase Stefan problem

Damian Słota *, Adam Zielonka

*Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland*

ARTICLE INFO

*Keywords:*
Stefan problem
Moving boundary problem
Variational iteration method

ABSTRACT

In this paper, we will use the variational iteration method to find an approximate solution of a one-phase Stefan problem. This problem consists in finding the distribution of temperature in the domain and the position of a moving interface. The problem under consideration is at first approximated with a system of differential equations in a domain with known boundary, and next, the system constructed in this way is solved by the variational iteration method. The validity of the approach is verified by comparing the results obtained with an analytical solution.



## 1. Introduction

The variational iteration method was devised by Ji-Huan He [1–4]. The method enables the determination of solutions of a wide range of non-linear operator equations

$$L(u(t)) + N(u(t)) = f(t), \tag{1}$$

where: $L$ is a linear operator, $N$ is a non-linear operator, $f(t)$ is a known function, whereas $u(t)$ a sought function. First, let us construct a correction functional

$$u_n(t) = u_{n-1}(t) + \int_0^t \lambda(s)\ \big(L(u_{n-1}(s)) + N(\tilde{u}_{n-1}(s)) - f(s)\big)\ \mathrm{d}s \tag{2}$$

where: $\tilde{u}_{n-1}$ is a restricted variation [1–7], $\lambda(s)$ is a general Lagrange multiplier [1,2,8], which can be optimally identified using the variational theory [1–3,9]. Next, on the grounds of the variational theory, the general Lagrange multiplier $\lambda(s)$ is determined. Finally, we obtain the iteration formula:

$$u_n(t) = u_{n-1}(t) + \int_0^t \lambda(s)\,(L(u_{n-1}(s)) + N(u_{n-1}(s)) - f(s))\,\mathrm{d}s, \tag{3}$$

from which, on the grounds of given initial approximation $u_0(t)$, an approximate solution (and frequently, an exact solution) of Eq. (1) may be derived.

The variational iteration method is useful for solving a wide range of problems [1–3,5–7,10–18]. The convergence of the method was discussed by Tatari and Dehghan [19]. Słota [20] applied the variational iteration method combined with optimization for the approximate solution of one-phase direct and inverse Stefan problems with a Dirichlet boundary condition. In Refs. [21,22] the Adomian decomposition method combined with optimization was used to arrive at an approximate solution of the Stefan problem; while in Ref. [23] the Adomian decomposition method was compared with the

* Corresponding author.
*E-mail addresses:* Damian.Slota@polsl.pl, d.slota@polsl.pl (D. Słota), Adam.Zielonka@polsl.pl (A. Zielonka).

0898-1221/$ – see front matter 
doi:10.1016/j.camwa.2009.03.070

Runge–Kutta method used to solve a system of ordinary non-linear differential equations derived from the transformations of the Stefan problem.

In the present study, we deal with using the variational iteration method for an approximate solution of a one-phase Stefan problem. A Stefan task is first approximated with a system of differential equations in the domain with a known boundary, and next, the system constructed in this way is solved by the variational iteration method. Owing to using this approach, we do not need to build a functional and find its minimum, as was the case in the paper [20].

## 2. The Stefan problem

The Stefan problem denotes mathematical models describing the thermal processes with phase change, during which heat is absorbed or emitted. Examples of such processes include: solidification of metals, freezing of water and soil, deep freezing of foodstuffs, melting of ice. The Stefan problem involves the designation of the temperature distribution in the domain and the position of the moving interface (freezing front), if the initial condition, boundary conditions and thermophysical properties of a physical body are known. A special case is a one-phase problem, where the temperature at one side of the moving interface is constant and equal to the temperature of the phase change. An example of such a case is the melting of ice, if it is under the melting temperature. In some simple cases, it is possible to find an analytical solution [24,25]. In all other cases, approximate methods must be employed [26,27].

Let $D = \{(x, t);\ t \in [0, t^*),\ x \in [0, \xi(t)]\}$ be a domain in $\mathbb{R}^2$. On particular parts:

$$\Gamma_0 = \{(x, 0);\ x \in [0, s],\ s = \xi(0)\}, \tag{4}$$

$$\Gamma_1 = \left\{(0, t);\ t \in [0, t^*)\right\}, \tag{5}$$

$$\Gamma_g = \left\{(x, t);\ t \in [0, t^*),\ x = \xi(t)\right\}, \tag{6}$$

of the domain boundary the initial and boundary conditions are met. Let us consider a one-phase one-dimensional Stefan problem in domain $D$. In such a case, we would like unknown functions $u(x, t)$ and $\xi(t)$ to comply with the following equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}(x,t), \quad \text{in } D, \tag{7}$$

$$u(x, 0) = \varphi(x), \quad \text{on } \Gamma_0, \tag{8}$$

$$\xi(0) = s, \tag{9}$$

$$u(0, t) = \theta(t), \quad \text{on } \Gamma_1, \tag{10}$$

$$u(\xi(t), t) = u^*, \quad \text{on } \Gamma_g, \tag{11}$$

$$k\,\frac{\partial u(x,t)}{\partial x} = \kappa\,\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}, \quad \text{on } \Gamma_g, \tag{12}$$

where $s > 0$, $\alpha$ is the thermal diffusivity, $k$ is the thermal conductivity, $\kappa$ is the latent heat of fusion per unit volume, $u^*$ is the temperature of the phase change, $u$ is temperature, and $t$ and $x$ refer to time and spatial location, respectively.

The solution of the Stefan problem formulated above involves the designation of the temperature distribution $u(x, t)$ and function $\xi(t)$ describing the moving interface position. In the problem considered, the remaining functions $\varphi(x)$, $\theta(t)$ and coefficients $\alpha$, $k$, $\kappa$, $u^*$ and $s$ are known.

## 3. Method of the solution

In the first step, let us reduce the formulated problem, designated in the curvilinear domain $D$, to a task designated in the domain of rectangular geometry. This may be performed by making the following substitution:

$$\begin{cases} y = \dfrac{x}{\xi(t)}, \\ \tau = t. \end{cases} \tag{13}$$

After this transformation, domain $D$ will change over to domain $\overline{D}$ (see Fig. 1), and boundaries $\Gamma_i$ to boundaries $\overline{\Gamma}_i$, $i \in \{0, 1, g\}$, where

$$\overline{D} = \left\{(y, \tau);\ \tau \in [0, t^*),\ y \in [0, 1]\right\},$$

and

$$\overline{\Gamma}_0 = \{(y, 0);\ y \in [0, 1]\},$$

$$\overline{\Gamma}_1 = \left\{(0, \tau);\ \tau \in [0, t^*)\right\},$$

$$\overline{\Gamma}_g = \left\{(1, \tau);\ \tau \in [0, t^*)\right\}.$$