Computers and Mathematics with Applications 71 (2016) 2155–2169

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A goal-oriented reduced-order modeling approach for nonlinear systems

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ARTICLE INFO

Article history:
Available online 10 March 2016

Keywords:
Reduced-order model
Goal-oriented
Proper orthogonal decomposition
Burgers equation
State estimator
Feedback control

ABSTRACT

In this paper, we develop a novel, goal-oriented reduced-order modeling methodology. The approach uses a low-dimensional basis function set that contains both global and local, goal-oriented basis functions. Compared to reduced-order models using the standard proper orthogonal decomposition (POD) basis, these new goal-oriented POD basis functions lead to better approximations of given quantities of interest (QoI) while maintaining accuracy in the evolution of the state. We demonstrate this approach for two problems involving Burgers equation. In the first problem, the QoI is the spatial average of the solution over various regions. The QoI in the second problem is the feedback control based on a MinMax control design with an extended Kalman filter. In both cases, approximations of the QoI and the state variables are more accurate using the goal-orientated POD than using the standard POD basis with comparable online computational costs.

1. Introduction

Repeated numerical simulations of large-scale, nonlinear dynamical systems are required in many engineering control and optimization problems. In addition, control laws based on compensators require the real-time simulation of nonlinear models that incorporate state measurements. Direct, full-order numerical simulations require large discretized systems for adequate approximation and are not feasible in many of these applications. Thus, reduced-order models (ROMs) based on the proper orthogonal decomposition (POD) combined with Galerkin projection [1] have been widely used to provide fast, accurate simulations of these large nonlinear systems.

Essentially, POD basis functions are solution-adapted basis functions that provide the *optimal* basis to represent a given set of simulation data or snapshots. In many cases, a handful of the leading POD modes can represent the most significant characteristics of the dynamical system, e.g., patterns in turbulent flows dominated by organized (coherent) structures. However, for highly non-stationary, nonlinear problems, POD-Galerkin models may lose their effectiveness. First of all, like the Fourier basis, POD modes are global; thus energy associated with each mode is distributed throughout the domain. As a result, one may have to use a large number of POD basis functions to accurately capture the energy transfer between modes, leading to increased computational cost. Secondly, the discarded POD modes can have a marked impact on the system, see [2]. Although they only hold a fractional share of the energy, their contribution to the dynamics of the retained modes can be significant. Therefore, the POD-ROM of complex systems can be unstable even when the POD basis retained in the ROM retains 99% of energy [3]. To obtain an accurate POD-ROM for complex systems, research has been done in two main

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http://dx.doi.org/10.1016/j.camwa.2016.01.031
0898-1221/

directions: (i) strategies to construct a more representative basis; (ii) modeling the effects of the discarded POD modes in the ROM. In this paper, we mainly focus on the first direction. Several methods have been proposed in the literature in this direction. They aim at improving the POD basis functions: (i) by the choice of locations of snapshots [4]; (ii) by the adjustment of weights on snapshots [5–7]; (iii) by the choice of inner product [7–9]; (iv) by the enrichment of the POD basis, e.g., to minimize the residual of the ROM [10–12], to fit certain physical phenomena [13], or to account for parametric changes [14]; and (v) by the choice of the time windows on which the POD method is performed [15,16]; etc.

In this paper, we develop a new basis selection strategy for the POD-ROM to overcome the degraded efficiency when applied to complex systems. The new approach is motivated by the following key observations: The POD basis is obtained by minimizing the time average projection error of the snapshots to the space spanned by the basis on the whole spatial domain in the entire time range. Therefore the leading POD basis functions represent the global features of the system well. This basis could fail to approximate the quantity of interest (e.g., the average value of a state variable in a particular subdomain) well and fails to display the features with small time scales (e.g. the high-frequency modes in fast transient flows). Therefore, to achieve an accurate representation of the system information, the ROM basis in the proposed approach consists of *global and local* modes. The global modes are the leading POD basis functions, while, the local modes are selected for subintervals of time in a goal-oriented way.

This novel method for local mode generation synthesizes ideas from the PID method in [15] and the goal-oriented approach proposed in [5]. By considering residuals of the discrepancies between the snapshots and their projection onto the subspace spanned by global modes, we seek local modes that maximize the contribution of the residuals to the quantities of interest over short (time) subintervals. Hence, they are able to capture important small temporal scales and, usually, small spatial scales that are missed in the standard POD approximation. This new basis selection strategy leads to new physical insights into the reduced-order basis and yields an efficient, reliable way to achieve ROMs for complex systems.

The remainder of the paper is organized as follows: the POD method is briefly introduced in Section 2; the new goal-oriented model reduction method is developed in Section 3; the optimization problem is discussed in Section 4; numerical examples illustrating the effectiveness of the proposed method are presented in Sections 5 and 6. Finally, we provide brief conclusions and directions of future work.

2. POD-Galerkin reduced-order models

The POD-Galerkin method to produce reduced-order models for nonlinear PDEs has two main steps, computation of the POD basis and construction of the low-dimensional dynamic model through Galerkin projection. In the first step, one precomputes simulation data that is representative of the behavior expected for the surrogate model. For the Burgers equation, e.g. [17], the data consists of m finite element solutions at uniform timesteps, referred to as simulation snapshots. The POD method seeks a low-dimensional basis that optimally approximates the snapshot data. Mathematically speaking, given rank d snapshot data from a Hilbert space $\mathcal{H}$, it chooses $\{\phi_1(x), \ldots, \phi_r(x)\} \subset \mathcal{H}$, for any $r \leq d$, to minimize the averaged projection error

$$\frac{1}{m}\sum_{\ell=1}^{m}\left\| w(\cdot,t_\ell) - \sum_{j=1}^{r}\left(w(\cdot,t_\ell),\phi_j(\cdot)\right)_{\mathcal{L}_2}\phi_j(\cdot)\right\|_{\mathcal{H}}^2 \tag{2.1}$$

subject to the conditions $(\phi_i,\phi_j)_{\mathcal{H}} = \delta_{ij},\ 1 \leq i,j \leq r$, where δ_{ij} is the Kronecker delta. Note that other inner product spaces can be readily implemented, but $\mathcal{H} = \mathcal{L}_2$ is typically used in practice. To solve (2.1), consider the following eigenvalue problem:

$$K\,\psi = \lambda\,\psi, \quad \text{where } K_{ij} = \frac{1}{m}\left(w(\cdot,t_i), w(\cdot,t_j)\right)_{\mathcal{H}} \tag{2.2}$$

for $1 \leq i,j \leq m$. The so-called temporal autocorrelation matrix $K \in \mathbb{R}^{m\times m}$ is symmetric, positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$, and corresponding (orthonormal) eigenvectors $\{\psi_j\}_{j=1}^{d}$. It can be shown [18–20] that the solution to (2.1) is given by

$$\phi_j(\cdot) = \frac{1}{\sqrt{\lambda_j}}\sum_{\ell=1}^{m}(\psi_j)_\ell\, w(\cdot,t_\ell), \quad 1 \leq j \leq r, \tag{2.3}$$

where $(\psi_j)_\ell$ is the ℓ-th component of the eigenvector ψ_j. The POD approximation of $w(x,t)$ is

$$w^r(x,t) \equiv \sum_{j=1}^{r}\phi_j(x)a_j(t), \tag{2.4}$$

where $\{a_j(t)\}_{j=1}^{r}$ are time-varying POD basis coefficient functions that must be determined. In most cases, r is chosen to be significantly smaller than both d and the number of spatial degrees of freedom (DOF) used to discretize the full-order model (FOM).