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# Feedback stabilization of a thermal fluid system with mixed boundary control

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ABSTRACT

We consider the problem of local exponential stabilization of the nonlinear Boussinesq equations with control acting on portion of the boundary. In particular, given a steady state solution on an bounded and connected domain $\Omega \subset \mathbb{R}^2$, we show that a finite number of controls acting on a part of the boundary through Neumann/Robin boundary conditions is sufficient to stabilize the full nonlinear equations in a neighborhood of this steady state solution. Dirichlet boundary conditions are imposed on the rest of the boundary. We prove that a stabilizing feedback control law can be obtained by solving a Linear Quadratic Regulator (LQR) problem for the linearized Boussinesq equations. Numerical result are provided for a 2D problem to illustrate the ideas.



## 1. Introduction

This paper is motivated by problems that occur naturally in modeling, designing and controlling energy efficient building systems. A standard problem of interest is concerned with controlling the indoor environment for both comfort (temperature, humidity) and air quality. Although computational fluid dynamics plays a huge role in much of this research, very little has been done on the mathematical and computational treatment of the distributed parameter control problems associated with such systems. We consider a boundary control problem governed by the Boussinesq equations that model heat transfer in a viscous incompressible heat-conducting fluid. The Boussinesq equations consist of the Navier–Stokes equations coupled to the convection–diffusion equation for temperature. This model assumes that the fluid has a uniform density.

We focus on a stabilization problem where the control inputs are limited to small portion of the boundary and the model system is governed by the Boussinesq equations on an open bounded and connected domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\Gamma$. The challenge arises from the stabilization of the Navier–Stokes equations and its coupling with the convection–diffusion equation for temperature. Controllability and stabilizability of the Navier–Stokes equations and Boussinesq equations have been widely discussed in [1–6]. The controllability results are mainly studied with the help of Carleman and observability inequalities, duality, and variational principles. Although Linear Quadratic Regulator (LQR) control design has been used to obtain stabilizing feedback controllers for the Navier–Stokes equations (see [7–14]), the majority of this work either assumes distributed control or Dirichlet boundary control. The most challenging case arises when using Dirichlet boundary control for Navier–Stokes equations in 3D. In particular, in [10,12] it is shown that in order to apply LQR theory to

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Navier–Stokes equations with Dirichlet boundary control, a *compatibility condition* needs to be satisfied in the 3D case. This condition has the form $\mathbf{v}(0)|_{\Gamma} = (\mathbf{v}|_{\Gamma})_{t=0} = u_{\mathbf{v}}(0)$, where $\mathbf{v}$ is the velocity field and $u_{\mathbf{v}} \in L^2(\Gamma)^d$ is the control input. Imposing this condition produces a rather complicated Riccati-like equation which is difficult to solve by standard numerical approaches (see [15]). In the paper [16] Badra considered a boundary control problem where Neumann boundary control acted on the entire boundary. Badra used a Lyapunov function method to stabilize the Boussinesq equations.

The specific problem of interest in this paper is a 2D problem with mixed boundary conditions, where the control inputs are of Neumann/Robin type and finite in number. In this setting, we employ a Riccati-based feedback controller to stabilize the nonlinear system. Moreover, under suitable conditions (see [17–21]) it is known that Robin boundary conditions can be used to approximate Dirichlet boundary conditions. We take advantage of this approximation to develop numerical algorithms that apply to Robin, Neumann, and Dirichlet boundary control problems. However, with mixed boundary conditions the solutions to the stationary Boussinesq equations lose regularity (see [22–24]) and this adds an additional difficulty.

Recently, Nguyen and Raymond in [25] also considered the stabilization problem Navier–Stokes equations with mixed boundary condition, where a localized Dirichlet boundary control is applied. In order to ensure regularity for the stationary problem, the authors assume that the junction between a segment with Neumann boundary condition and a segment with Dirichlet boundary condition forms a right angle. This regularity vanishes as the angle approaches $\pi$ which occurs in the room problem because of the geometry of the room. This issue will be discussed fully in Section 2.2.

The outline of this paper is as follows. We first discuss the stabilizability of the linearized Boussinesq equations with finite dimensional controllers and set up the LQR control problem for the system. Sufficient conditions are given for stabilizability of the linearized system for the case where the steady state solution is unstable. An LQR problem is employed to generate a linear feedback operator by solving a suitable Riccati operator equation. Then we show that the semigroup generated by the linearized closed-loop system operator is exponentially stable and apply a fixed point theorem to prove that the controlled nonlinear Boussinesq system is locally exponentially stable. Finally, we provide numerical results for a 2D room model by employing a penalized finite element method. The numerical results illustrate the theoretical results and provide some indication of the effectiveness of the LQR designed feedback control.

## 2. Model description

In order to focus the ideas, consider the 2D version of a heated room shown in Fig. 1. Assume that the airflow is coming in through the inlet vent, denoted by $\Gamma_{\mathrm{I}}$, which is defined as a portion of the boundary $\Gamma$, with Robin boundary control for both velocity and temperature. The airflow exits at the vent defined by the boundary $\Gamma_{\mathrm{O}}$ with stress-free fluid and natural (or unforced) convective flux boundary conditions. In addition, let $\Gamma_{\mathrm{H}}$ denote the radiant heating strip on the floor with Neumann boundary control for temperature. We impose no slip boundary conditions for the velocity on $\Gamma_{\mathrm{f}} = \Gamma \setminus (\Gamma_{\mathrm{I}} \cup \Gamma_{\mathrm{O}})$ and zero Dirichlet boundary condition for temperature on $\Gamma_{\mathrm{D}} = \Gamma \setminus (\Gamma_{\mathrm{I}} \cup \Gamma_{\mathrm{O}} \cup \Gamma_{\mathrm{H}})$. Note that the boundaries $\Gamma_{\mathrm{I}}$, $\Gamma_{\mathrm{O}}$ and $\Gamma_{\mathrm{H}}$ are disjoint. This simple zone configuration is typical of the systems of interest and will be used to illustrate the theoretical and numerical results developed below. We turn now to the more general case.

### 2.1. Boussinesq equations with mixed boundary conditions

Assume $\Omega \subseteq \mathbb{R}^2$ is an open bounded and connected domain with a regular boundary $\Gamma$. The Boussinesq model is given by

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}, = \frac{1}{Re}\Delta\mathbf{v} - \nabla p + \bar{\mathbf{e}}\frac{G_r}{Re^2}\theta + f_{\mathbf{v}} \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{v}\cdot\nabla\theta = \frac{1}{RePr}\Delta\theta + f_{\theta} \quad \text{in } \Omega, \tag{2.3}$$

where $\mathbf{v}(x,t)$ is the velocity, $p(x,t)$ is the pressure, $\theta(x,t)$ is the fluid temperature, $Re$ is the Reynolds number, $Gr$ is the Grashof number, $Pr$ is the Prandtl number, and $\bar{\mathbf{e}} = [0, 1]^{\mathrm{T}}$ is the gravitational force direction. We assume that $f_{\mathbf{v}}$ is a time independent external body force and $f_{\theta}$ is a time independent heat source density.

In order to set up the abstract formulation for the mixed boundary conditions, we introduce the divergence free spaces for velocity

$$\mathbf{H} = \{\mathbf{v} \in (L^2(\Omega))^2 : \operatorname{div}\mathbf{v} = 0, \mathbf{v}\cdot n|_{\Gamma_{\mathrm{f}}} = 0\},$$

where $n$ denotes the outward unit normal vector with respect to the domain $\Omega$,

$$\mathbf{V}^s = (H^s(\Omega))^2 \cap \mathbf{H}, \quad s \in \mathbb{R},$$

and

$$\mathbf{V}^s_{\Gamma_{\mathrm{f}}} = \{\mathbf{v} \in (H^s(\Omega))^2 : \operatorname{div}\mathbf{v} = 0, \mathbf{v}|_{\Gamma_{\mathrm{f}}} = 0\}, \quad s > 1/2.$$