Computers and Mathematics with Applications 71 (2016) 2257–2271

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Mass-conservative Fourier spectral methods for solving the fractional nonlinear Schrödinger equation☆

Siwei Duo, Yanzhi Zhang*

Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA

ARTICLE INFO

Article history:
Available online 3 February 2016

Keywords:
Fractional nonlinear Schrödinger equation
Fractional Laplacian
Split-step method
Crank–Nicolson method
Relaxation method

ABSTRACT

We propose three Fourier spectral methods, i.e., the split-step Fourier spectral (SSFS), the Crank–Nicolson Fourier spectral (CNFS), and the relaxation Fourier spectral (ReFS) methods, for solving the fractional nonlinear Schrödinger (NLS) equation. All of them are mass conservative and time reversible, and they have the spectral order accuracy in space and the second-order accuracy in time. In addition, the CNFS and ReFS methods are energy conservative. The performance of these methods in simulating the plane wave and soliton dynamics is discussed. The SSFS method preserves the dispersion relation, and thus it is more accurate for studying the long-time behaviors of the plane wave solutions. Furthermore, our numerical simulations suggest that the SSFS method is better in solving the defocusing NLS, but the CNFS and ReFS methods are more effective for the focusing NLS.

1. Introduction

The fractional Schrödinger equation is a nonlocal dispersive equation, which was first introduced in [1,2] by generalizing the Feynman path integral over Lévy trajectories. Recently, it has also been derived as the continuum limit of a microscopic lattice system with long-range interactions [3]. The nonlocality of the fractional Schrödinger equation enables it to describe new phenomena which are absent from the standard Schrödinger equation [4–7]. However, the nonlocality also introduces considerable challenges in finding the solutions of the fractional Schrödinger equation. Hence, the understanding of its solutions still remains limited. In this paper, we propose three mass-conservative Fourier spectral methods for numerically solving the fractional nonlinear Schrödinger (NLS) equation, and the performance of these methods is examined and compared both analytically and numerically.

We consider the fractional nonlinear Schrödinger (NLS) equation in the semiclassical regime [8–10]:

$$i\varepsilon \frac{\partial u(\mathbf{x}, t)}{\partial t} = \varepsilon^{\alpha}(-\Delta)^{\alpha/2} u(\mathbf{x}, t) + \beta |u(\mathbf{x}, t)|^{2\sigma} u(\mathbf{x}, t), \quad t > 0, \tag{1.1}$$

$$u(\mathbf{x}, 0) = \psi(\mathbf{x}), \tag{1.2}$$

where $i = \sqrt{-1}$, and $u(\mathbf{x}, t)$ is a complex-valued wave function of $\mathbf{x} \in \mathbb{R}^d$ (for $d = 1, 2$, or 3) and $t \geq 0$. $0 < \varepsilon \leq 1$ is a small semiclassical parameter, and the constant $\sigma > 0$. The parameter $\beta \in \mathbb{R}$ describes the strength of short-range (or

☆ This research was supported by the National Science Foundation grant DMS-1217000.

* Corresponding author.
E-mail addresses: sddy9@mst.edu (S. Duo), zhangyanz@mst.edu (Y. Zhang).

http://dx.doi.org/10.1016/j.camwa.2015.12.042
0898-1221/

local) nonlinear interactions; if β is positive (resp. negative), the interactions are repulsive or defocusing (resp. attractive or focusing). The fractional Laplacian $-(-\Delta)^{\alpha/2}$ is defined via a pseudo-differential operator [11–14]:

$$-(-\Delta)^{\alpha/2}u(\mathbf{x}) := \mathcal{F}^{-1}\left[-|\boldsymbol{\xi}|^{\alpha}\mathcal{F}[u]\right], \quad \alpha > 0, \tag{1.3}$$

where $\mathcal{F}$ represents the Fourier transform, and $\mathcal{F}^{-1}$ denotes its inverse. If $\alpha = 2$, the fractional Laplacian in (1.3) reduces to the standard Laplace operator, but for $\alpha \in (0, 2)$, it is a nonlocal operator describing long-range interactions [11,3,15].

Similar to the standard ($\alpha = 2$) NLS, the fractional NLS (1.1) has two important conserved quantities: the *mass* of the wave function:

$$N(t) = \|u(\cdot, t)\|^2 := \int_{\mathbb{R}^d} |u(\mathbf{x}, t)|^2 d\mathbf{x} \equiv N(0), \tag{1.4}$$

and the *total energy* (or *Hamiltonian*):

$$E(t) = \int_{\mathbb{R}^d} \left[\varepsilon^{\alpha}\mathrm{Re}\Big(u^*(\mathbf{x}, t)(-\Delta)^{\alpha/2}u(\mathbf{x}, t)\Big) + \frac{\beta}{\sigma+1}|u(\mathbf{x}, t)|^{2(\sigma+1)}\right] d\mathbf{x} \equiv E(0), \tag{1.5}$$

where $\mathrm{Re}(\phi)$ and ϕ^*, respectively, represent the real part and the complex conjugate of a function ϕ. The fractional NLS is *time reversible*, that is, (1.1) remains invariant if one replaces the time t by $-t$ and takes its conjugate. These properties are usually used as benchmarks to develop and examine numerical methods for the fractional NLS.

The fractional NLS (1.1) admits the plane wave solution of the form:

$$u(\mathbf{x}, t) = A\exp\big[i(\boldsymbol{\lambda}_{\mathbf{k}} \cdot \mathbf{x} - \omega t)\big], \tag{1.6}$$

provided that the *dispersion relation*

$$\omega = \varepsilon^{\alpha-1}|\boldsymbol{\lambda}_{\mathbf{k}}|^{\alpha} + \frac{\beta}{\varepsilon}|A|^{2\sigma}, \tag{1.7}$$

is satisfied. Here, A is the amplitude of the plane wave solution, $\boldsymbol{\lambda}_{\mathbf{k}} \in \mathbb{R}^d$ is the vector of wave numbers, and ω is the time frequency. It shows in [15] that due to its nonlocality, the stability and dynamics of the plane wave solutions in the fractional NLS are significantly different from those in the standard NLS. In numerical studies of plane wave dynamics, it is desirable for a numerical method to preserve the dispersion relation in (1.7).

Both the nonlocality and nonlinearity of the fractional NLS make it extremely challenging to find its analytical solutions. Therefore, numerical simulations play an important role in the study of the fractional NLS. However, in contrast to the standard NLS, only a few numerical methods are available in the literature for solving the fractional NLS. For example, a second-order finite difference method is proposed in [16,17], which is based on the fractional centered difference discretization of the Riesz fractional derivatives. In [18], collocation methods are presented to solve the fractional linear Schrödinger equation. Recently, a split-step Fourier spectral method is applied in [19] to study the decoherence of solitons in the fractional Schrödinger equation, but the properties of this method are not discussed. In this paper, we aim to develop accurate numerical methods for solving the fractional NLS, which preserve one or more analytical properties of the fractional NLS, including mass conservation, energy conservation, time reversible, and dispersion relation. In addition, some remarks are provided for numerical simulations of the fractional NLS.

The paper is organized as follows. In Section 2, we introduce three methods for solving the fractional NLS, and the properties of these methods are analyzed in detail. Numerical examples are presented in Section 3 to examine and compare the performance of our numerical methods in solving the fractional NLS. Some concluding remarks are made in Section 4.

2. Numerical methods

In this section, we introduce three Fourier spectral methods for solving the fractional NLS (1.1)–(1.2) and prove that all of them are mass conservative in the discrete level. Some other properties, such as energy conservation, time reversible, and dispersion relation, are also discussed for each method. For simplicity of notation, we will introduce our numerical methods for the one-dimensional (i.e., $d = 1$) fractional NLS. The generalization of these methods to higher dimensions is straightforward. First, we truncate (1.1)–(1.2) into a finite computational domain $[-L, L]$ with periodic boundary conditions and consider the following problem:

$$i\varepsilon\partial_t u(x, t) = \varepsilon^{\alpha}(-\Delta)^{\alpha/2}u(x, t) + \beta|u(x, t)|^{2\sigma}u(x, t), \quad t > 0, \tag{2.1}$$

$$u(x, 0) = \psi(x), \tag{2.2}$$

for $x \in [-L, L]$. We usually choose L to be sufficiently large, unless plane wave solutions are studied [15]. We will leave the discussion of other boundary conditions for our future work.

Let $\tau > 0$ denote a time step, and define the time sequence $t_n = n\tau$ for $n \geq 0$. Define the mesh size $h = 2L/J$, with J a positive even integer. Denote spatial grid points $x_j = -L + jh$, for $0 \leq j \leq J$. Let u_j^n be the numerical approximation of the solution $u(x_j, t_n)$. Then, we denote the solution vector at time $t = t_n$ as $U^n = (u_0^n, u_1^n, \ldots, u_{J-1}^n)^T$. Due to the definition of