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A spectral mimetic least-squares method for the Stokes equations with no-slip boundary condition

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ABSTRACT

Formulation of locally conservative least-squares finite element methods (LSFEMs) for the Stokes equations with the no-slip boundary condition has been a long standing problem. Existing LSFEMs that yield exactly divergence free velocities require non-standard boundary conditions (Bochev and Gunzburger, 2009 [3]), while methods that admit the no-slip condition satisfy the incompressibility equation only approximately (Bochev and Gunzburger, 2009 [4, Chapter 7]). Here we address this problem by proving a new non-standard stability bound for the velocity–vorticity–pressure Stokes system augmented with a no-slip boundary condition. This bound gives rise to a norm-equivalent least-squares functional in which the velocity can be approximated by div-conforming finite element spaces, thereby enabling a locally-conservative approximations of this variable. We also provide a practical realization of the new LSFEM using high-order spectral mimetic finite element spaces (Kreeft et al., 2011) and report several numerical tests, which confirm its mimetic properties.

1. Introduction

In this paper we consider least-squares finite element methods (LSFEMs) for the velocity–vorticity–pressure (VVP) formulation of the Stokes problem

$$\begin{cases} \nabla \times \boldsymbol{\omega} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \nabla \times \boldsymbol{u} - \boldsymbol{\omega} = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\boldsymbol{u}$ denotes the velocity, $\boldsymbol{\omega}$ the vorticity, p the pressure and $\boldsymbol{f}$ the force per unit mass. Our main focus is on the formulation of conforming LSFEMs that are (i) locally conservative, and (ii) provably stable when the system (1) is augmented with the no-slip (velocity) boundary condition

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Note that (2) is equivalent to a pair of boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

for the normal and tangential components of the velocity field, respectively.

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Two factors motivate the choice of (1) as a foundation for our method. Using first-order systems has been a staple in least-squares formulations because it allows one to reduce the regularity requirement on the finite element spaces as well as the condition number of the resulting algebraic problems. A second consideration is the practical importance of the vorticity variable in applications where rotational flow dominates the flow dynamics, such as rotor aerodynamics, the flow around wind turbines or wake flows. Methods that can directly control and possibly reduce the error in the vorticity can be of significant value in these applications.

Formulation of conforming LSFEMs that satisfy both (i) and (ii) had been a long-standing challenge. Existing conforming least-squares methods generally fall into one of the following two categories. The LSFEMs in the first category, see e.g., [1,2], are stable and accurate for (1) with the boundary condition (2) but satisfy $\nabla\cdot\boldsymbol{u} = 0$ only approximately. Conversely, the LSFEMs in the second category; see, e.g., [3], [4, Chapter 7] yield exactly divergence free velocity fields but require the non-standard normal velocity, tangential vorticity boundary condition

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{\omega}\times\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{4}$$

i.e., they specify only the first of the two velocity conditions in (3).

Thus far, achieving both stability and exact mass conservation with the velocity boundary condition has been only possible by switching to non-conforming formulations such as the discontinuous LSFEMs in [5,6], or by employing Lagrange multipliers to enforce mass conservation [7]. Of course the latter negates some of the attractive properties of least-squares methods such as symmetric and positive definite algebraic systems, while the former requires careful selection of mesh-dependent weights for the various jump terms and can result in higher condition numbers. In either case, the resulting least-squares methods tend to be less attractive computationally. It should be mentioned that mass conservation in least-squares methods can be strengthened although not satisfied exactly by employing an additional weight for the residual of the continuity equation [8]. Although this approach can partially mitigate the loss of mass conservation it tends to increase the condition number of the resulting system and to reduce the accuracy with which the formulation satisfies the rest of the equations.

In this paper we address formulation of conservative LSFEMs for the VVP Stokes system with (2) by developing a new, non-standard a priory stability bound for this problem. We refer to this bound as "non-standard" because (i) it uses a "weaker" L^2-norm to measure the residual of the momentum equation in (1), instead of a conventional Sobolev space norm, and (ii) it employs a weak curl in the second equation of (1) and a weak grad operator in the first equation of (1). This stability bound gives rise to a norm-equivalent functional, which can be discretized by using div-conforming elements for the velocity field. We show that the resulting LSFEM is both locally conservative and stable for (1)–(2).

We have organized the rest of the paper as follows. Section 2 introduces notation and some necessary background results. Section 3 establishes a non-standard stability bound for the VVP system and Section 4 presents the associated least-squares formulation. Section 5 describes the compatible (mimetic) spectral element spaces, which we use to discretize the least-squares principle and explains how all necessary differential operators can be expressed by operations on the associated degrees-of-freedom. We present numerical results in Section 6 and conclude with some remarks in Section 7.

2. Preliminaries

In what follows $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded open region with Lipschitz boundary $\Gamma = \partial\Omega$. We recall the space $L^2(\Omega)$ of all square integrable functions with norm and inner product denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$, respectively, and its subspace $L^2_0(\Omega)$ of all square integrable functions with a vanishing mean. The spaces $H(\text{curl},\Omega)$ and $H(\text{div},\Omega)$ contain square integrable functions whose curl and divergence are also square integrable. When equipped with the graph norms

$$\|\boldsymbol{\xi}\|^2_{\text{curl}} := \|\boldsymbol{\xi}\|^2_0 + \|\nabla\times\boldsymbol{\xi}\|^2_0 \quad \text{and} \quad \|\boldsymbol{v}\|^2_{\text{div}} := \|\boldsymbol{v}\|^2_0 + \|\nabla\cdot\boldsymbol{v}\|^2_0,$$

the spaces $H(\text{curl},\Omega)$ and $H(\text{div},\Omega)$ are Hilbert spaces. We recall the subspaces

$$H_0(\text{curl},\Omega) = \{\boldsymbol{v}\in H(\text{curl},\Omega) \mid \boldsymbol{v}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$
$$H_0(\text{div},\Omega) = \{\boldsymbol{v}\in H(\text{div},\Omega) \mid \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

of $H(\text{curl},\Omega)$ and $H(\text{div},\Omega)$, respectively containing functions whose tangential and normal traces vanish on the boundary.

2.1. Adjoint operators

The mimetic least-squares method in this paper requires two additional operators acting on $L^2(\Omega)$ and $H(\text{div},\Omega)$ functions. Their definition follows.

Definition 1. The adjoint gradient of $p\in L^2(\Omega)$ is the function $\nabla^* p \in H(\text{div},\Omega)$, which satisfies the relation

$$(\nabla^* p, \boldsymbol{v})_0 := (p, -\nabla\cdot\boldsymbol{v})_0 + \int_{\partial\Omega} p(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}S, \quad \forall \boldsymbol{v}\in H(\text{div},\Omega). \tag{5}$$

The adjoint gradient defines a map $\nabla^* : L^2(\Omega) \mapsto H(\text{div},\Omega)$.