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A finite element method for computing accurate solutions for Poisson equations with corner singularities using the stress intensity factor

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ABSTRACT

In this article, we consider the Poisson equation with homogeneous Dirichlet boundary conditions, on a polygonal domain with one reentrant corner. The solution of the Poisson equation with a concave corner yields a singular decomposition, $u = w + \lambda\eta s$, where w is regular, s is a singular function, and the coefficient λ is the so called stress intensity factor. This stress intensity factor can be computed using the extraction formula. We introduce a new non-homogeneous boundary value problem, which has 'zero' stress intensity factor. Using the solution of this new partial differential equation, we can compute an accurate solution of the original problem, simply by adding singular part. We obtain an optimal convergence rate with smaller errors when compared with others.

1. Introduction

Let Ω be an open, bounded polygonal domain in $\mathbb{R}^2$, and let Γ be the boundary of the domain Ω. For a given function $f \in L^2(\Omega)$, we consider the following Poisson equation with homogeneous Dirichlet boundary conditions as a model problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where Δ denotes the Laplacian operator.

If the domain is convex or smooth, we expect to achieve an optimal convergence rate using the standard finite element method for such elliptic boundary value problems, with respect to the mesh size h. However, this is not true for Poisson problems defined on non-convex domains.

Assume that Ω is a polygonal domain with one reentrant corner. Then, the solution of (1.1) exhibits singular behavior near that corner, even when f is very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain. Without loss of generality, we can assume that the reentrant corner is at the origin.

Over recent decades, many researchers have proposed various methods for overcoming this phenomenon. Many have approached this by manipulating the meshes, while others have suggested augmenting the trial and/or the test spaces with some singular or dual singular functions (e.g., [1,2]). The singular function method (SFM) and the dual singular function

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method (DSFM) are well known examples of the second type of approach (e.g., [3,4,2,5–11]). Adaptive mesh refinement methods have also been investigated very actively, in [1,12–15] and references therein.

The precise forms of the singular solutions of many partial differential equations are well known, for example for the Poisson equation, interface problem, or elastic equation [2,16–18]. If we let ω denote the internal angle of Ω satisfying $\pi < \omega < 2\pi$, then the singular function s and its dual singular function s_- are defined by

$$s = s(r,\theta) = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_- = s_-(r,\theta) = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega} \tag{1.2}$$

for the model problem (1.1), and the unique solution $u \in H_0^1(\Omega)$ has the representation (see [16])

$$u = w + \lambda\eta s, \tag{1.3}$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and η is a smooth cut-off function that equals one identically in a neighborhood of the origin, with the support of η small enough such that the function ηs vanishes identically on Γ ((r,θ) are polar coordinates).

The coefficient λ is called the 'stress intensity factor', and can be computed by the following extraction formula (see [16]):

$$\lambda = \frac{1}{\pi}\int_\Omega f\eta s_- dx + \frac{1}{\pi}\int_\Omega u\Delta(\eta s_-)dx. \tag{1.4}$$

Note that both s and s_- are harmonic functions in Ω.

Some numerical approaches (e.g., [2,19,20]) use this extraction formula for λ, and seek the regular part $w \in H^2(\Omega) \cap H_0^1(\Omega)$ from a new partial differential equation, such as

$$-\Delta w = f + \lambda\Delta(\eta s) \quad \text{in } \Omega. \tag{1.5}$$

Unfortunately, these results were not good enough because the input function f is replaced by $f + \lambda\Delta(\eta s)$, etc., whose L^2-norms are quite large compared to that of f (see Lemma 2.2). To overcome this drawback, we convert this homogeneous boundary value problem to a non-homogeneous one without using a cut-function η.

In this paper, we introduce a new non-homogeneous partial differential equation, whose solution is in $H^2(\Omega)$ with the same input function f, through a simple change of the boundary condition. Using this partial differential equation, we propose an efficient algorithm for computing a numerical solution for the Poisson equation with a singular domain.

We propose the following solution procedure:

Step (1) Find the solution u from the partial differential equation (1.1), using the standard finite element method.
Step (2) Compute the stress intensity factor λ using the extraction formula.
Step (3) Pose the new partial differential equation, which has zero stress intensity factor, and find the solution w, such that

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = -\lambda s|_\Gamma & \text{on } \Gamma. \end{cases} \tag{1.6}$$

Step (4) Set $u = w + \lambda s$.

A couple of examples will be given in Section 4, where computational results are provided using FreeFEM++ code.

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$, and standard associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, with their respective norms and seminorms denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively. However, we will omit Ω if there is no chance of misunderstanding. Furthermore, we denote $H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$.

2. New algorithm and theoretical results

We use a cut-off function to isolate the singular behavior of the problem. Therefore, we will first give the definition of the cut-off functions with parameters. Set

$$B(r_1;r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega \quad \text{and} \quad B(r_1) = B(0;r_1).$$

2.1. Cut-off functions

We define a family of cut-off functions of r as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B\left(\frac{1}{2}\rho\right), \\ \frac{15}{16}\left\{\frac{8}{15} - p(r) + \frac{2}{3}p(r)^3 - \frac{1}{5}p(r)^5\right\} & \text{in } \bar{B}\left(\frac{1}{2}\rho;\ \rho\right), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho), \end{cases} \tag{2.1}$$

where $p(r) = 4r/\rho - 3$. Here, ρ is a parameter that will be determined such that the singular part $\eta_\rho s$ has the same boundary condition as the solution u of the model problem, where s is the singular function given in (1.2). Note that $\eta_\rho(r)$ is C^2.