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Algorithms and models for turbulence not at statistical equilibrium

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ABSTRACT

Standard eddy viscosity models, while robust, cannot represent backscatter and have severe difficulties with complex turbulence not at statistical equilibrium. This report gives a new derivation of eddy viscosity models from an equation for the evolution of variance in a turbulent flow. The new derivation also shows how to correct eddy viscosity models. The report proves the corrected models preserve important features of the true Reynolds stresses. It gives algorithms for their discretization including a minimally invasive modular step to adapt an eddy viscosity code to the extended models. A numerical test is given with standard and over diffusive eddy viscosity models. The correction (scaled by 10^{-8}) does exhibit intermittent backscatter.

1. Introduction

The sensitivity and richness of scales in turbulent flows motivate many approaches to modeling turbulence. Fundamentally, they involve choosing an averaging operator, separating variables into means and fluctuations, deriving non-closed equations for means and modeling the effect of fluctuations on means. One fundamental approach is statistical modeling which begins with ensemble averaging. Usually the ergodic hypothesis is invoked and ensemble averaging is converted into time averaging, leading to a collection of RANS and URANS models of increasing complexity. Local spacial averaging and increasing computing power has led to reinterpretation of early, simple statistical models as LES models (identifying the mixing length with the averaging radius). The boundaries between these (and other) approaches overlap considerably.

With increased computational resources, solving for a moderately sized velocity ensemble begins to be feasible and is required to quantify uncertainty. Thus, ensemble averaging and statistical models (without invoking ergodicity) become possible and are considered herein. Most[1] statistical models are based on eddy viscosity (EV) with differences in how the eddy viscosity coefficient is determined from the flow variables. Due to the wide experience with them, their limitations are also well recognized. EV models represent only dissipative effects of the Reynolds stresses and cannot represent intermittent energy flow from turbulent fluctuations back to the mean velocity[2] without ad hoc fixes, such as negative viscosities ("*more than a little strange*" in [2, Section 5.9, p. 373]). Based on an exact equation derived in Section 2 linking this energy flow to

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1 For example, only Chapter 6 of [1] addressed non-eddy viscosity models.

2 This flow is called backscatter for local spacial averaging. It occurs also for statistical models and URANS models. For RANS models based on long time averaging the result of Chacon-Rebollo and Lewandowski, Remark 2.4, confirms that it does not occur in a space averaged sense.

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the Reynolds stresses, Section 3 shows how to correct any eddy viscosity models systematically to include this intermittent energy flow from fluctuations back to means. The proposed correction is based on a new and fundamental derivation of eddy viscosity models. Various aspects of the corrected models are explored in Sections 4–7.

1.1. Formulation

To begin, given an ensemble of initial conditions

$$u(x, 0; \omega_j) = u_0(x; \omega_j), \quad j = 1, \dots, J, \ x \in \Omega,$$

let $u(x, t; \omega_j)$, $p(x, t; \omega_j)$ be associated solutions to the incompressible Navier–Stokes equations (NSE) driven by a body force $f(x, t)$

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f(x, t), \quad \text{and} \quad \nabla \cdot u = 0, \quad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Let $\langle \cdot \rangle$ denote ensemble averaging

$$\langle u \rangle (x, t) := \frac{1}{J} \sum_{j=1}^{J} u(x, t; \omega_j) \quad \text{and} \quad u'(x, t; \omega_j) = u(x, t; \omega_j) - \langle u \rangle (x, t).$$

Ensemble averaging the NSE yields the non-closed system: $\nabla \cdot \langle u \rangle = 0$ and

$$\langle u \rangle_t + \langle u \rangle \cdot \nabla \langle u \rangle - \nu \triangle \langle u \rangle - \nabla \cdot R(u, u) + \nabla \langle p \rangle = f(x, t), \tag{1.2}$$

where the Reynolds stress $R(u, u)$, e.g., [1, Section 2.4], [3, Section 3.2], [2, Section 5.1], is

$$R(u, u) := \langle u \rangle \otimes \langle u \rangle - \langle u \otimes u \rangle = -\left\langle u' \otimes u' \right\rangle.$$

Statistical models of turbulence (based on the Boussinesq assumption) begin with ensemble averaging and replace $R(u, u)$ by an enhanced viscous term depending only on $\langle u \rangle$. Based on the material of Section 2, Section 3 shows that these eddy viscosity models are based on three steps.

1. The Boussinesq assumption (from [4,5] and Theorem 2.3) that turbulent fluctuations (the action of $\nabla \cdot R(u, u)$ in (1.2)) are dissipative on average in (1.2). This is followed by assuming that space–time *averaged* dissipativity holds *pointwise* in time and space.

2. The eddy viscosity hypothesize that this dissipativity aligns with the gradient or deformation tensor and thus can be represented by a viscous term with a turbulent viscosity coefficient $\nu_T(\langle u \rangle)$, [6].

3. Model parameterization/calibration is done by fitting the turbulent viscosity coefficient $\nu_T(\langle u \rangle)$ to flow data. Calibration is equivalent to specifying a fluctuation model for $\nabla u'$ in terms of $\nabla \langle u \rangle$.

The resulting eddy viscosity model (whose solution $w(x, t)$ with pressure $q(x, t)$ is intended to be an approximation of the true average velocity $\langle u \rangle$) results: $\nabla \cdot w = 0$ and

$$w_t + w \cdot \nabla w - \nabla \cdot ([\nu + \nu_T(w)] \nabla w) + \nabla q = f(x, t), \quad \text{in } \Omega, \tag{EV}$$
$$w = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad w(x, 0) = \langle u_0 \rangle \quad \text{in } \Omega.$$

EV models are the models of choice for most industrial turbulent flows[3] and many increasingly complex parametrizations of $\nu_T(w)$ are known, e.g., [1,3,7–18]. They have well recognized limitations in not modeling complex turbulence, backscatter or turbulence not at statistical equilibrium, e.g., [19–22]. (The second assumption that the dissipativity of the Reynolds stress term aligns with $\nabla \langle u \rangle$ also fails for some flows, [20], but is not the issue addressed herein.)

The correction required for eddy viscosity models to represent backscatter in non-statistically stationary turbulence, the case when the action of the fluctuations is intermittently non-dissipative, is derived and analyzed herein. Given the eddy viscosity parameterization $\nu_T(w)$, choose a re-scaling parameter $\beta > 0$ and define

$$a(w) := \sqrt{\nu^{-1}\nu_T(w)}.$$

The corrected EV model (derived in Section 2) is then $\nabla \cdot w = 0$ and

$$w_t + \beta^2 a(w) \frac{\partial}{\partial t} (a(w)w) + w \cdot \nabla w - \nabla \cdot ([\nu + \nu_T(w)] \nabla w) + \nabla q = f(x, t). \tag{Corrected EV}$$

In Section 3 time averaged dissipativity, an important feature of the true Reynolds stresses, is proven to be preserved in (Corrected EV).

The (Corrected EV) differs from (EV) by the extra term $\beta^2 a(w) \frac{\partial}{\partial t} (a(w)w)$. This term means time discretization introduces new issues, especially in adapting legacy codes from (EV) to (Corrected EV). Section 4 shows how time discretization can be done and preserve these important model properties, including the important case of modular adaptation of a legacy code available for (EV). Phenomenology is used to obtain some insight into calibration of the re-scaling parameter β in Section 7. Section 8 tests corrections of a standard mixing length model and the over-diffused Smagorinsky model. Even with a very small re-scaling of the correction, $\beta^2 \simeq O(10^{-8})$, the numerical test shows that the corrected model does exhibit backscatter.

[3] *"Virtually all practical engineering computations are done with some variety of eddy viscosity formulation..."* [3, Section 7.4.1, p. 195].