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A least-squares finite element method for a nonlinear Stokes problem in glaciology

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ABSTRACT

A stationary Stokes problem with nonlinear rheology and with mixed no-slip and sliding basal boundary conditions is considered. The model describes the flow of ice in glaciers and ice sheets. A least-squares finite element method is developed and analyzed. The method does not require that the finite element spaces satisfy an inf–sup condition. Moreover, the usage of negative Sobolev norm in the least-squares functional allows for the use of standard piecewise polynomials spaces for both the velocity and pressure approximations. A Picard-type iterative method is used to linearize the Stokes problem. It is shown that the linearized least-squares functional is coercive and continuous in an appropriate solution space so the existence and uniqueness of a weak solution immediately follows as do optimal error estimates for finite element approximations. Numerical tests are provided to illustrate the theory.

1. Introduction

Ice sheets such as Greenland and Antarctica and glaciers are massive ice forms formed from snow and layers of frozen water that accumulate over long periods of time. They move very slowly, either descending from mountains or moving outward from a center of accumulation. Several models have been suggested and theoretical and numerical analyses for glacier and ice-sheet evolution had been studied; see, e.g., [1–8]. One such model is given by the nonlinear Stokes problem

$$\begin{cases} -\nabla\cdot\big(\mu(\mathbf{u})(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\big)+\nabla p=\mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ and p denote the velocity and pressure fields, respectively, and $\mu(\mathbf{u})$ denotes a velocity-dependent viscosity coefficient whose exact form is given in Section 3. In [9,10], similar fluid equations with homogeneous Dirichlet boundary condition are analyzed. However, at the ice–bedrock interface, ice flows of glaciers and ice sheets satisfy a Robin-type condition; see [11]. In [2], the well-posedness of (1.1) was established for a particular form of boundary conditions of this type. In [12], a Galerkin finite element method was used to determine approximations of solutions of (1.1). There, several means to stabilize Galerkin finite element method were addressed (see also [13,14]) and enhancements that preserve the stability of approximate solutions are discussed.

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In this paper, we also impose the Rayleigh friction boundary condition used in [12] at the ice–bedrock boundary but, for discretization purposes, instead use a least-squares finite element method. The method is based on first recasting (1.1) into an equivalent first-order system of partial differential equations an then simply minimizing their residuals in a least-squares sense. This approach has been successfully used in many applications [15–19] due to its merits that include discrete linear systems that are symmetric positive definite, circumvention of LBB-type stability conditions so that the use of equal-order finite elements spaces for velocity and pressure approximations is allowed, least-squares functionals providing explicit local a posteriori error measures, applicability to concurrent simulations of multiple physics, and so on. In particular, we propose a minimization problem that involves a least-squares functional that is defined using somewhat non-standard norms. The existence, uniqueness, and finite element approximations of solutions of the least-square problem are explored and numerical test results are presented.

In Section 2, useful definitions, notations, and well-known theorems are reviewed. In Section 3, the nonlinear Stokes equations with boundary conditions appropriate to the glaciology setting are introduced. In Section 4, an iterative method based on a Picard-type linearization is presented as a means for solving the nonlinear problem. The linear problems faced at each iteration are then solved using a least-squares minimization problem, as presented in Section 5 where also the existence and uniqueness of the corresponding minimizer is proved. In Section 6, finite element approximations are analyzed and then illustrative numerical results are presented in Section 7.

2. Preliminaries

We review some useful definitions, notations, and well-known theorems and lemmas. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, denote a bounded open connected domain with Lipschitz boundary Γ. The standard Sobolev spaces for functions defined on Ω and Γ are denoted by $H^r(\Omega)$ and $H^r(\Gamma)$ with associated norms $\|\cdot\|_r$ and $\|\cdot\|_{r,\Gamma}$, respectively, where r is a real number. Vector- and tensor-valued functions are written in bold characters. Inner products in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_\Gamma$, respectively. For the sake of notational simplicity, subscripts for L^2-norms are omitted, e.g., as in $\|\cdot\|$.

For $\Gamma_f \subset \Gamma$, let $H^1_f(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_f\}$; of course, $H^1_0(\Omega)$ corresponds to $\Gamma_f = \Gamma$. Let $H^{-1}_f(\Omega)$ denote the dual space of $H^1_f(\Omega)$ equipped with the norm

$$\|u\|_{H^{-1}_f(\Omega)} = \sup_{v\in H^1_f(\Omega), v\neq 0} \frac{\langle u, v\rangle}{\|v\|_1},$$

which, for notational simplicity, we simply refer to as $\|\cdot\|_{-1}$. Define

$$H(\nabla\cdot) = \{\mathbf{u} \in [L^2(\Omega)]^d \mid \nabla\cdot\mathbf{u} \in L^2(\Omega)\}$$

which is a Hilbert space under the norm

$$\|\mathbf{u}\|_{H(\nabla\cdot)} = (\|\mathbf{u}\|^2 + \|\nabla\cdot\mathbf{u}\|^2)^{\frac{1}{2}}.$$

Throughout this paper, c and C denote generic constants whose value may be different at different instances.

Several well-known theorems and lemmas are used in this paper. The proof of the first lemma follows along the lines of that in [20,21] for $[H^1_0(\Omega)]^d$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ denote a bounded connected domain with Lipschitz boundary and let $V^\perp$ denote the H^1-orthogonal complement of*

$$V := \{\boldsymbol{\varphi} \in [H^1_f(\Omega)]^d \mid \langle\nabla\cdot\boldsymbol{\varphi}, \zeta\rangle = 0 \text{ for all } \zeta \in L^2(\Omega)\}.$$

Then the mapping $\nabla\cdot : V^\perp \to L^2(\Omega)$ is an isomorphism. Moreover, for any $q \in L^2(\Omega)$, there exist a function $\mathbf{v} \in V^\perp \subset [H^1_f(\Omega)]^d$ and a constant c depending on Ω but not on $\mathbf{v}$ satisfying

$$\nabla\cdot\mathbf{v} = q \quad \text{and} \quad \|\mathbf{v}\|_1 \le c\|q\|.$$

The following generalized Korn inequality can be found in [22]. The second Korn inequality is a special case of the generalized inequality [20].

Theorem 2.1 (*Generalized Korn Inequality*)**.** *Let $\Omega \subset \mathbb{R}^d$ denote a bounded Lipschitz domain and let $\mathbf{u} \in [H^1(\Omega)]^d$. Then there exists a constant $c > 0$ such that*

$$\|\mathbf{u}\|_1 \le c\left(\|\mathbf{u}\| + \|\nabla\mathbf{u} + \nabla\mathbf{u}^T\|\right).$$

Theorem 2.2 (*Second Korn Inequality*)**.** *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^d$ and let $\mathbf{u} \in [H^1(\Omega)]^d$. In addition, suppose that $\mathbf{u}$ vanishes on some part of the boundary Γ that has positive $(d-1)$ dimensional measure. Then there exists a positive constant c such that*

$$\|\mathbf{u}\|_1 \le c\left\|\nabla\mathbf{u} + \nabla\mathbf{u}^T\right\|. \tag{2.1}$$