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A fourth-order compact solution of the two-dimensional modified anomalous fractional sub-diffusion equation with a nonlinear source term

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ABSTRACT

This work is concerned to the study of high order difference scheme for the solution of a two-dimensional modified anomalous sub-diffusion equation with a nonlinear source term which describes processes that become less anomalous as time progresses. The space fractional derivatives are described in the Riemann–Liouville sense. In the proposed scheme we discretize the space derivatives with a fourth-order compact scheme and use the Grünwald–Letnikov discretization of the Riemann–Liouville derivatives to obtain a fully discrete implicit scheme. We prove the stability and convergence of proposed scheme using the Fourier analysis. The convergence order of the proposed method is $\mathcal{O}(\tau + h_x^4 + h_y^4)$. Comparison of numerical results with analytical solutions demonstrates the unconditional stability and high accuracy of proposed scheme.

1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [1–4]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering [5]. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most applications are given in the book of Oldham and Spanier [6], the book of Podulbny [4] and the papers of Metzler and Klafter [7], Bagley and Torvik [8]. Many considerable works on the theoretical analysis [9,10] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly. So many authors have resorted to numerical solution strategies based on convergence and stability analysis [5,11–18]. Liu has carried on so many works on the finite difference method for the fractional partial differential equations [19–21]. The homotopy analysis method in [22,23] is applied to solve some linear and nonlinear fractional partial differential equations. Authors of [24] proposed the Sinc functions and Legendre polynomial techniques to reduce the fractional convection–diffusion equation with variable coefficients to the solution of system of linear algebraic equations. In [25] a general formulation for the Legendre operational matrix of fractional derivative has been derived. The main aim of [26] is to apply a technique based on the shifted Legendre-tau idea to solve the fractional diffusion equations with variable coefficients on a finite domain. A numerical method for solving the linear and nonlinear fractional integro-differential equations of Volterra type is presented in [27]. Also using compact finite difference for the solution of some one-dimensional fractional partial differential equations is introduced in [28,29].

There are several definitions of a fractional derivative of order $\alpha > 0$ [3,6]. The two most commonly used are the Riemann–Liouville and Caputo. The difference between the two definitions is in the order of evaluation [30]. As mentioned

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in [20,31] in numerous physical and biological systems, many diffusion rates of species cannot be characterized by the single parameter of the diffusion constant. Instead, the (anomalous) diffusion is characterized by a scaling parameter as well as a diffusion constant K, and the mean square displacement of diffusing species $\langle x^2(t)\rangle$ scales as a nonlinear power-law in time [20]

$$\left\langle x^2(t)\right\rangle \sim \frac{2K_\gamma}{\Gamma(1+\gamma)}t^\gamma, \quad t\to\infty,$$

where γ (with $0<\gamma<1$) is the anomalous diffusion exponent and K_γ is the generalized diffusion coefficient. For example, single particle tracking experiments and photo-bleaching recovery experiments have revealed sub-diffusion ($0<\gamma<1$) of proteins and lipids in a variety of cell membranes [32–38]. Anomalous sub-diffusion has also been observed in neural cell adhesion molecules [20,39]. Recently, models have been proposed to describe processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator with a nonlinear source term [20,40]

$$\frac{\partial u(x,t)}{\partial t}=\left(\mathcal{A}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}+\mathcal{B}\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]+f(u(x,t),x,t), \quad 0<x<L,\ 0<t\le T. \tag{1.1}$$

Liu et al. [40] proposed a semi-discrete approximation and a full discrete finite element approximation for the modified anomalous sub-diffusion equation (1.1) in a finite domain. They proved the stability and convergence of proposed methods. Authors of [20] proposed a conditionally stable difference scheme for the solution of (1.1). They show that the convergence order of method is $\mathcal{O}(\tau+h^2)$ with energy method. The aim of this paper is to propose an unconditionally stable difference scheme of order $\mathcal{O}(\tau+h_x^4+h_y^4)$ for the solution of following two-dimensional modified anomalous sub-diffusion equation with a nonlinear source term

$$\frac{\partial u(x,y,t)}{\partial t}=\left(\mathcal{A}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}+\mathcal{B}\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}\right]+g(u,x,y,t),$$
$$0<x<L,\ 0<y<L,\ 0<t\le T, \tag{1.2}$$

and boundary conditions

$$\begin{aligned} &u(0,y,t)=\varphi_1(y,t), \qquad u(L,y,t)=\varphi_2(y,t), \quad 0\le y\le L,\ 0<t\le T,\\ &u(x,0,t)=\psi_1(x,t), \qquad u(x,L,t)=\psi_2(x,t), \quad 0\le x\le L,\ 0<t\le T, \end{aligned} \tag{1.3}$$

and initial condition

$$u(x,y,0)=\phi(x,y), \quad 0\le x\le L,\ 0\le y\le L, \tag{1.4}$$

where $0<\alpha<1$, $0<\beta<1$ and $\mathcal{A}$, $\mathcal{B}$ are positive constants and the nonlinear source term has the first order continuous partial derivative $\frac{\partial g(u,x,y,t)}{\partial t}$. The symbols $\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}$ and $\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}$ are the Riemann–Liouville fractional derivative operator and are defined as

$$\frac{\partial^{1-\alpha}u(x,t)}{\partial t^{1-\alpha}}={}_0D_t^{1-\alpha}u(x,t)=\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{u(x,\eta)}{(t-\eta)^{1-\alpha}}d\eta,$$
$$\frac{\partial^{1-\beta}u(x,t)}{\partial t^{1-\beta}}={}_0D_t^{1-\beta}u(x,t)=\frac{1}{\Gamma(\beta)}\frac{\partial}{\partial t}\int_0^t\frac{u(x,\eta)}{(t-\eta)^{1-\beta}}d\eta,$$

where $\Gamma(\cdot)$ is the gamma function. Also, let $g(u,x,y,t)$ satisfies the Lipschitz condition with respect to u, i.e.

$$|g(\overline{u},x,y,t)-g(\widetilde{u},x,y,t)|\le\mathcal{L}\,|\overline{u}-\widetilde{u}|, \quad \forall\,\overline{u},\ \widetilde{u},$$

where $\mathcal{L}$ is a Lipschitz constant. For the solution of Eq. (1.2) we will apply a fourth-order difference scheme for discretizing the spatial derivatives and Grünwald–Letnikov discretization for the Riemann–Liouville fractional derivatives. Also we will discuss the stability of the proposed method by the Fourier method and show that the compact finite difference scheme converges with the spatial accuracy of fourth-order. The outline of this paper is as follows. In Section 2, we introduce the derivation of new method for the solution of Eq. (1.2). This scheme is based on approximating the time derivative of the mentioned equation by a scheme of order $\mathcal{O}(\tau)$ and spatial derivatives with a fourth order compact finite difference scheme. In Section 3 we prove the unconditional stability property of the method using the Fourier method. In Section 4 we present the convergence of the method and show that the convergence order is $\mathcal{O}(\tau+h_x^4+h_y^4)$. The numerical experiments of solving Eq. (1.2) with the method developed in this paper for several test problems are given in Section 5. Finally concluding remarks are drawn in Section 6.