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# Multiplicity of positive radial solutions for the weighted $p$-Laplacian in $\mathbb{R}^n \setminus \{0\}$ ☆

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ABSTRACT

This paper is concerned with the multiplicity of positive radial solutions for the weighted $p$-Laplacian in $\mathbb{R}^n \setminus \{0\}$. By constructing suitable Banach spaces, special cones and combining with the Avery–Peterson fixed point theorem, we obtain the existence of at least three positive radial solutions.



## 1. Introduction

In this paper, we study the multiplicity for the weighted $p$-Laplacian

$$-\operatorname{div}(\lambda(|x|)|\nabla u|^{p-2}\nabla u) = \lambda(|x|)\phi(|x|)h(x, u, \nabla u), \quad x \in \mathbb{R}^n \setminus \{0\} \tag{1.1}$$

where $p > 1, n \geq 2,\ \lambda(t) \in C^1(0, +\infty)$ with $\lambda(t) > 0$ on $(0, +\infty)$, $\phi(t)$ is a nonnegative measurable function defined on $(0, +\infty)$, both $\phi(t)$ and $\lambda(t)$ may be singular at $t = 0$, $h(x, u, \xi)$ is continuous, may be singular at $x = 0$ and has a radially symmetric structure.

The existence and multiplicity of positive solutions to the $p$-Laplacian together with Dirichlet, Sturm–Liouville or nonlinear boundary value conditions have been investigated extensively in recent years (see [1–9] and the references therein). As far as we know, most of the papers are for the equations when the nonlinear term $h$ does not explicitly involve the gradient of the solution. For example, we refer to [1] for the investigation of the existence of positive radial solutions for the $p$-Laplacian with singular sources which are allowed to change sign in a ball. In [2,3] the authors considered the existence and multiplicity of positive radial solutions for the $p$-Laplacian boundary value problems, and it was Chan-Gyun Kim, Eun Kyoung Lee, Yong-Hoon Lee [4] who studied the existence, nonexistence, and multiplicity of positive radial solutions for the $p$-Laplacian with boundary parameters on the exterior domain of a ball, using a combination of a fixed point theorem, the method of upper and lower solutions, and fixed point index theory in the frame of the ordinary differential equation technique. On the other hand, many mathematical workers researched the existence and multiplicity of positive solutions to the one-dimensional $p$-Laplacian, see [5–9].

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Our aim here is to find the positive radial solutions of Eq. (1.1) in $\mathbb{R}^n \setminus \{0\}$, which may have singularities at the isolated singular point $x = 0$, just as pointed by Véron [10]. For this purpose, we need to prescribe some suitable boundary conditions at 0 and $\infty$, which can be approximated by the boundary conditions on the boundary of the annular domain $\{x \in \mathbb{R}^n : R_1 < |x| < R_2\}$, where $0 < R_1 < R_2 < \infty$. Precisely speaking, we are interested in one of the three pairs of boundary conditions:

$$\lim_{x\to 0} u(x) + g_1\left(\varphi_p^{-1}\left(\lim_{x\to 0} |x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\right)\right) = 0,$$

$$\lim_{|x|\to +\infty} u(x) + g_2\left(\varphi_p^{-1}\left(\lim_{|x|\to +\infty} |x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\right)\right) = 0, \tag{1.1$_a$}$$

$$\lim_{x\to 0} |x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0,$$

$$\lim_{|x|\to +\infty} |u(x)|^{p-2}u(x) + \delta^{p-1}\lim_{|x|\to +\infty} |x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \tag{1.1$_b$}$$

$$\lim_{x\to 0} |u(x)|^{p-2}u(x) + \beta^{p-1}\lim_{x\to 0} |x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0,$$

$$\lim_{|x|\to +\infty} |x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \tag{1.1$_c$}$$

which can be obtained by letting $R_1 \to 0^+$ and $R_2 \to +\infty$ in the following boundary conditions respectively

$$u(x) + g_1\left(\varphi_p^{-1}\left(|x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\right)\right) = 0, \quad \text{on } |x| = R_1,$$

$$u(x) + g_2\left(\varphi_p^{-1}\left(|x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\right)\right) = 0, \quad \text{on } |x| = R_2,$$

$$|x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \quad \text{on } |x| = R_1,$$

$$|u(x)|^{p-2}u(x) + \delta^{p-1}|x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \quad \text{on } |x| = R_2,$$

$$|u(x)|^{p-2}u(x) + \beta^{p-1}|x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \quad \text{on } |x| = R_1,$$

$$|x|^{n-1}\lambda(|x|)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \quad \text{on } |x| = R_2,$$

where $\varphi_p^{-1}(t)$ is the inverse of $\varphi_p(t)$, $\varphi_p(t) = |t|^{p-2}t$, $g_1(t)$ and $g_2(t)$ are continuous odd functions defined on $\mathbb{R}$, $\partial u/\partial \nu$ denotes the normal derivative at the boundary, $\beta, \delta \geq 0$. From the assumption (H4) below, we see that the above-mentioned boundary conditions include the Dirichlet boundary conditions, such as (1.1$_a$) with $g_1(t) = g_2(t) \equiv 0$, the mixed boundary conditions, such as (1.1$_b$) with $\delta = 0$ and (1.1$_c$) with $\beta = 0$, and the Robin boundary conditions, such as (1.1$_a$) with $g_1(t), g_2(t)$ being both proportional functions [11], (1.1$_b$) with $\delta > 0$ and (1.1$_c$) with $\beta > 0$ [12,13]. Further, let $u(t) = u(|x|)$ with $t = |x|$ be a radially symmetric solution. Then a direct calculation shows that

$$(\omega(t)\varphi_p(u'(t)))' + \eta(t)f(t, u(t), u'(t)) = 0, \quad 0 < t < +\infty \tag{1.2}$$

with one of the three pairs of boundary conditions:

$$\lim_{t\to 0^+} u(t) - g_1\left(\lim_{t\to 0^+}(\varphi_p^{-1}(\omega)u')(t)\right) = 0, \qquad \lim_{t\to +\infty} u(t) + g_2\left(\lim_{t\to +\infty}(\varphi_p^{-1}(\omega)u')(t)\right) = 0, \tag{1.2$_a$}$$

$$\lim_{t\to 0^+}(\varphi_p^{-1}(\omega)u')(t) = 0, \qquad \lim_{t\to +\infty} u(t) + \delta\lim_{t\to +\infty}(\varphi_p^{-1}(\omega)u')(t) = 0, \tag{1.2$_b$}$$

$$\lim_{t\to 0^+} u(t) - \beta\lim_{t\to 0^+}(\varphi_p^{-1}(\omega)u')(t) = 0, \qquad \lim_{t\to +\infty}(\varphi_p^{-1}(\omega)u')(t) = 0, \tag{1.2$_c$}$$

where $\omega(t) = \lambda(t)t^{n-1}$, $\eta(t) = \omega(t)\phi(t)$ and $f(t, u, u') = h(x, u, \nabla u)$.