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Spectral methods for the time fractional diffusion–wave equation in a semi-infinite channel

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ABSTRACT

In this paper, we consider the numerical approximation of the time fractional diffusion–wave equation in a semi-infinite channel. The time fractional derivative is described in Caputo sense with order γ $(1 < \gamma < 2)$. A fully discrete spectral scheme based on a finite difference method in the time direction and a Laguerre–Legendre spectral method in the space direction is proposed. We also propose an alternating direction implicit (ADI) spectral scheme in order to reduce the amount of computation. The stability and convergence of both schemes are rigorously established. Numerical results are presented to support our theoretical analysis.

1. Introduction

Fractional calculus has a history as long as the classical calculus [1]. But only in the past decades does the fractional calculus undergo a rapid development. Readers can refer to [2] for an extensive list of recent applications and mathematical developments of the fractional calculus. Fractional differential equations can be used to describe lots of phenomena in physics, economics, engineering, chemistry, biology and other sciences [3]. Such as anomalous diffusion [4], relaxation and reaction kinetics of polymers [5], image processing [6], bioengineering [7], continuous-time finance [8] and so on.

For numerical approximation of the fractional differential equations, one may use finite difference method [9,10], finite element method [11,12], discontinuous Galerkin methods [13,14], spectral methods [15,16], etc. Numerical methods for solving time fractional partial differential equations have been considered by many authors [17–20]. In these papers, the stability and convergence are also discussed.

It is also interesting and important to consider the numerical approximation of the fractional differential equations in unbounded domains. In some previous investigations for these problems, an artificial boundary is introduced. Gao et al. [21] proposed a finite difference scheme for fractional sub-diffusion equations on a half line by using artificial boundary conditions; Brunner et al. [22] derived exact boundary conditions for a time fractional diffusion–wave equation on a two-dimensional unbounded spatial domain, and used finite difference method to solve the reduced problem on bounded computational domain.

In this paper we consider the time fractional diffusion–wave equation of the form

$$ {}_0^C D_t^{\gamma} u(x, y, t) = \Delta u(x, y, t) + g(x, y, t), \quad (x, y) \in \Omega,\ 0 < t \le T \tag{1} $$

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subject to the following initial and boundary conditions:

$$u(x,y,0)=\Phi(x,y),\qquad \frac{\partial u}{\partial t}(x,y,0)=\Psi(x,y),\quad (x,y)\in\Omega, \tag{2}$$

$$u|_{\partial\Omega}=0,\qquad \lim_{x\to+\infty}u=0,\quad 0\le t\le T, \tag{3}$$

where

$${}_0^C D_t^{\gamma}u(x,y,t)=\frac{1}{\Gamma(2-\gamma)}\int_0^t \frac{\partial^2 u(x,y,s)}{\partial s^2}\frac{ds}{(t-s)^{\gamma-1}},\quad 1<\gamma<2 \tag{4}$$

is the Caputo fractional derivative of order γ with respect to time t, Δ is the two-dimensional Laplacian, $\Omega=(0,+\infty)\times(-1,1)$, $\partial\Omega$ is the boundary of the domain Ω, $\Phi(x,y)$, $\Psi(x,y)$ and $g(x,y,t)$ are given functions.

Many authors considered this problem in bounded domains. Zhang et al. [23] proposed a compact alternating direction implicit (ADI) scheme whose order is $\mathcal{O}(\tau^{3-\gamma}+h_1^4+h_2^4)$, where τ is the temporal grid size and h_1,h_2 are spatial grid sizes in the x and y directions, respectively; Wang and Vong [24] proposed a high order ADI scheme with an order $\mathcal{O}(\tau^2+h_1^2+h_2^4)$; Li et al. [25] proposed an ADI Galerkin finite element method for the two-dimensional fractional diffusion–wave equation and analysed its stability and convergence.

However, the investigation of the problem in unbounded domain is sparse. In this paper, we use a mixed Laguerre–Legendre spectral method to solve the problem (1)–(3) numerically. We first transform the initial equation (1) into its equivalent integro-differential form with a Riemann–Liouville fractional integral operator, then we propose a fully discrete spectral scheme based on a finite difference method in the time direction and a spectral Galerkin approximation with Laguerre functions in the x direction and Legendre polynomials in the y direction. More precisely, we use a weighted Grünwald difference operator to discretize the Riemann–Liouville integral operator, then based on a Crank–Nicolson technique, the convergence rate of the fully discrete scheme in L^2 norm is $\mathcal{O}(\tau^2+M^{(1-r)/2}+N^{1-m})$, where τ is temporal step size, M,N are two positive integers as discretized parameters in x and y space directions, and r,m are the regularity of the exact solution with respect to x and y, respectively. We give a detailed analysis for the stability and convergence of the fully discrete scheme. We then propose an ADI spectral scheme which can significantly reduce the computation time and storage requirements, and its stability and convergence are analysed.

The rest of the paper is organized as follows. In Section 2, some preliminaries and notations are shown, including a mixed projection approximation operator which is essential in the proof of the convergence of the fully discrete scheme. In Section 3, we present the formulation of the fully discrete spectral scheme. The stability and convergence of the fully discrete scheme are analysed in Section 4. In Section 5, we propose an ADI spectral method, and analyse its stability and convergence. We do some numerical experiments in Section 6. Finally, some conclusions are given in Section 7.

2. Preliminaries and notations

We first give some notations and projection approximation results in the x direction. Let $\mathbb{R}^+=(0,+\infty)$, and $\chi(x)$ be a certain weight function. $L^2_\chi(\mathbb{R}^+):=\{v|v$ is measurable and $\int_{R^+}v^2\chi dx<\infty\}$, and its inner product $(\cdot,\cdot)_{\chi,\mathbb{R}^+}$ and norm $\|\cdot\|_{\chi,\mathbb{R}^+}$ are defined by

$$(u,v)_{\chi,\mathbb{R}^+}=\int_{R^+}uv\chi dx,\qquad \|u\|_{\chi,\mathbb{R}^+}=(u,u)^{\frac{1}{2}}_{\chi,\mathbb{R}^+}.$$

We drop the subscript χ in the previous notations when $\chi(x)\equiv 1$.

We define $H^1(\mathbb{R}^+)=\{v|v\in L^2(\mathbb{R}^+),\ v_x\in L^2(\mathbb{R}^+)\}$ equipped with the following norm and semi-norm

$$\|u\|_{1,\mathbb{R}^+}=(\|u\|^2_{\mathbb{R}^+}+\|u_x\|^2_{\mathbb{R}^+})^{\frac{1}{2}},\qquad |u|_{1,\mathbb{R}^+}=\|u_x\|_{\mathbb{R}^+}.$$

$H^1_0(\mathbb{R}^+):=\{v|v\in H^1(\mathbb{R}^+),\ v(0)=0\}$.

For simplicity, we denote $\partial_x^k v(x)=\frac{d^k}{dx^k}v(x)$. Throughout the paper, c denotes a generic positive constant.

Let $\mathcal{L}_n(x)$ be the Laguerre polynomial of degree n, M be a positive integer; we denote the Laguerre function by

$$\widehat{\mathcal{L}}_i(x)=\mathcal{L}_i(x)e^{-x/2},$$

and set

$$\widehat{\mathbb{P}}_M=\mathrm{span}\{\widehat{\mathcal{L}}_i(x),\ i=0,1,\dots,M\},$$

$\widehat{\mathbb{P}}^0_M=\{\phi\in\widehat{\mathbb{P}}_M:\phi(0)=0\}$.

Let $\widehat{\partial}_x=\partial_x+1/2$, $\widehat{\omega}^\alpha=x^\alpha$, define $\widehat{B}^r(\mathbb{R}^+)=\{u:\widehat{\partial}_x^k u\in L^2_{\widehat{\omega}_k}(\mathbb{R}^+),\ 0\le k\le r\}$ equipped with the norm and semi-norm

$$\|u\|_{\widehat{B}^r,\mathbb{R}^+}=\left(\sum_{k=0}^r\|\widehat{\partial}_x^k u\|^2_{\widehat{\omega}_k,\mathbb{R}^+}\right)^{1/2},\qquad |u|_{\widehat{B}^r,\mathbb{R}^+}=\|\widehat{\partial}_x^r u\|_{\widehat{\omega}_r,\mathbb{R}^+}.$$