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Multigrid methods for saddle point systems using constrained smoothers

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ABSTRACT

The constrained smoother for solving the saddle point system arising from the constrained minimization problem is a relaxation scheme such that the iteration remains in the constrained subspace. A multigrid method using constrained smoothers for saddle point systems is analyzed in this paper. Uniform convergence of two-level and W-cycle multigrid methods, with sufficient many smoothing steps and full regularity assumptions, are obtained for some stable finite element discretization of Stokes equations. For Braess–Sarazin smoother, a convergence theory using only partial regularity assumption is also developed.

1. Introduction

Due to their indefiniteness and poor spectral properties, saddle point problems are difficult to solve. Multigrid (MG) methods, one of the most efficient solvers for symmetric positive definite problems, work less efficiently for saddle point problems. In this paper, we shall design and analyze effective multigrid methods for the saddle point problems:

$$\begin{pmatrix} A & B^T \\ B & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{1}$$

where A is a symmetric and positive definite (SPD) operator and B is surjective. Eq. (1) arises from mixed finite element methods discretization of partial differential equations (PDEs), notably the Stokes equations in fluid dynamics in which $A = -\Delta$, and $B = -\mathrm{div}$.

The main difficulty of developing robust and effective multigrid methods for the saddle point system (1) is due to the constraint $Bu = 0$. Recall that the success of multigrid method relies on two ingredients: the high frequency can be damped efficiently by the smoother, and the low frequency can be well approximated by the coarse grid correction. For saddle point systems, however, both smoothing and coarse grid correction can easily violate the constraint.

We propose to use the constrained smoother which is defined as a relaxation scheme such that the iteration remains in the constrained subspace $\mathcal{K} = \ker(B)$. For Stokes equations, this means that the velocity iteration is always divergence free. To derive constrained smoothers, we introduce the operator $A_{\mathcal{K}} = Q_{\mathcal{K}} A Q_{\mathcal{K}}^T : \mathcal{K} \to \mathcal{K}$, where $Q_{\mathcal{K}} = I - B^T(BB^T)^{-1}B$ is the L^2-projection to $\mathcal{K}$, and rewrite the saddle point system (1) as a symmetric positive definite equation

$$A_{\mathcal{K}} u = Q_{\mathcal{K}} f. \tag{2}$$

We shall design smoothers for (1) based on smoothers for (2). The operator $A_{\mathcal{K}}$ is introduced for the theoretical propose and will not be formed explicitly. Namely the algorithm we derived will involve only components of the original saddle point system.

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We shall show that Richardson iteration for solving (2) is the Braess–Sarazin (B–S) smoother [1] for (1), and Jacobi and Gauss–Seidel iterations for (2) correspond to additive and multiplicative Schwarz smoothers considered in Schöberl [2], which is better known as Vanka smoother [3] in the context of computational fluid dynamics.

For the coarse grid correction, the difficulty is that the constrained subspaces in consecutive levels are non-nested. To overcome this non-nestedness, we propose to use a L^2-type projection $\mathbb{Q}_{\mathcal{K}}$ to bring the coarse grid correction back to $\mathcal{K}$. One generic choice is $\mathbb{Q}_{\mathcal{K}} = Q_{\mathcal{K}}$ which requires a Poisson type solver. When the pressure space consists of discontinuous elements, following Schöberl [2], we can choose a localized L^2 projection by using elements in the coarse grid as a non-overlapping domain decomposition of the underlying domain. We would like to mention that when constrained subspaces are nested, a multilevel method based on the constrained energy minimization and its convergence analysis has been developed recently in [4].

It has been numerically observed that multiplicative Schwarz smoother leads to an efficient multigrid methods for saddle point problems, however, theoretical analysis for the convergence is only available for less efficient additive versions [2,5]. One contribution of this paper is to extend the smoothing property of the additive Schwarz smoother established by Schöberl [2] to the multiplicative Schwarz smoothers.

With the smoothing property and the approximation property, we are able to prove that the two-level method and W-cycle multigrid method using constrained smoothers converge uniformly provided the full regularity assumption of Stokes equations and the assumption of sufficiently many smoothing steps.

Another contribution of this paper is to present a multigrid convergence proof without the full regularity assumption. For scalar elliptic equations, the MG theory has undergone stages of development from regularity based multigrid theory [6] to regularity free (or less) one [7–12]. Surprisingly enough the current MG theory for saddle point problems is still in the full regularity stage [13,1,14–16]. Only very recently, Brenner, Li, and Sung [17] developed new multigrid methods for Stokes equations and have proved the uniform convergence without the full regularity assumption. Our smoother and consequently the convergence analysis is different with that in [17].

We shall follow Bank and Dupont [7] to present a convergence proof using only partial regularity assumption of the Stokes equation. Consequently our analysis can be applied to more realistic problems especially for solutions with singularity. We verify the approximation and smoothing property using an operator dependent norm for the Braess–Sarazin smoother. We shall also follow Bramble, Pasciak, and Xu [18] to use the variable V-cycle multigrid as a preconditioner which can relax the assumption of sufficiently many smoothing steps.

We are aware that more effective block preconditioners for the Stokes equations are available [19,20]. The analysis here is of theoretical interest since the convergence of multigrid methods for Stokes equations with the partial regularity assumption is rare.

The rest of this paper is structured as follows. In Section 2, we present the setting of the problem including notation and different formulations of the saddle point system. In Section 3, we introduce constrained relaxation schemes and in Section 4, we present the two-level method and W-cycle multigrid and prove their uniform convergence. In Section 5, we verify the smoothing and approximation property for Vanka smoothers and in Section 6, we establish the convergence theory with partial regularity assumption when using Braess–Sarazin smoother. We refer the reader to [1,2] for numerical results that are consistent and supporting our theoretical results.

2. Problem setting

Let $\mathcal{H}$ be a Hilbert space equipped with inner product $(\cdot,\cdot)$ and $\mathcal{V} \subset \mathcal{H}$ be a closed subspace. Suppose $A : \mathcal{V} \to \mathcal{V}$ is a symmetric and positive definite (SPD) operator with respect to $(\cdot,\cdot)$, which introduces a new inner product $(u, v)_A := (Au, v) = (u, Av)$ on $\mathcal{V}$. The norm associated to $(\cdot,\cdot)$ or $(\cdot,\cdot)_A$ will be denoted by $\|\cdot\|$ or $\|\cdot\|_A$, respectively. Let $\mathcal{P}$ be another Hilbert space and let $B : \mathcal{V} \to \mathcal{P}$ be a linear operator continuous in $\|\cdot\|_A$. With a slight abuse of notation, we still denote the inner product of $\mathcal{P}$ by $(\cdot,\cdot)$. In most problems of consideration, the inner product $(\cdot,\cdot)$ for $\mathcal{H}$ is the vector L^2-inner product while for $\mathcal{P}$ it is the scalar L^2-inner product.

We are interested in solving the following saddle point system: For a given $f \in \mathcal{H}$, find $u \in \mathcal{V}$, $p \in \mathcal{P}$ such that

$$\begin{aligned} (Au, v) + (p, Bv) &= (f, v) && \text{for all } v \in \mathcal{V}, \\ (Bu, q) &= 0 && \text{for all } q \in \mathcal{P}, \end{aligned}$$

which will be written in the operator form

$$\begin{pmatrix} A & B^T \\ B & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{3}$$

The operator matrix in (3) will be abbreviated as $\mathcal{L} = (A, B^T; B, O)$ and Eq. (3) can be written as $\mathcal{L}(u, p) = (f, 0)$. Here $(\cdot)^T$ is the adjoint with respect to the default inner product $(\cdot,\cdot)$ and a functional in the dual space $\mathcal{H}'$ is identified as an element in $\mathcal{H}$ through the Riesz map induced by $(\cdot,\cdot)$. Throughout this paper, we assume the well-posedness of (3) and focus on its efficient solvers.

We shall consider geometric multigrid methods for solving the saddle point problem (3) which arises from mixed finite element method discretizations of elliptic partial differential equations. A typical and important example is the finite