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On coupled Navier–Stokes and energy equations in exterior-like domains

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ABSTRACT

We investigate a class of nonlinear evolution systems modeling time-dependent flows of incompressible, viscous and heat-conducting fluids with temperature dependent transport coefficients in three-dimensional exterior-like domains. We prove a local existence theorem for the fully coupled parabolic system with a source term involving the square of the velocity gradient and a combination of Dirichlet and artificial boundary conditions.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. Suppose that Γ_D and Γ_N are closed disjoint two-dimensional manifolds of class $C^{1,1}$ such that $\partial\Omega = \Gamma_D \cup \Gamma_N$. Γ_D represents solid surfaces and Γ_N denotes the artificial part of the boundary $\partial\Omega$. Let $T \in (0, \infty)$ be fixed throughout the paper, $I = (0, T)$ and $\Omega_T = \Omega \times I$, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$. The strong formulation of our problem reads as follows:

$$\varrho\left(\boldsymbol{u}_t + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})\right) - \nabla\cdot(\nu(\theta)\mathbb{D}(\boldsymbol{u})) + \nabla\pi = \boldsymbol{F}(\theta) \quad \text{in } \Omega_T, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega_T, \tag{1.2}$$

$$\varrho c_v\left(\theta_t + \boldsymbol{u}\cdot\nabla\theta\right) - \nabla\cdot(\lambda(\theta)\nabla\theta) - \nu(\theta)\mathbb{D}(\boldsymbol{u}):\mathbb{D}(\boldsymbol{u}) = 0 \quad \text{in } \Omega_T, \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{DT}, \tag{1.4}$$

$$\theta = \theta_D \quad \text{on } \Gamma_{DT}, \tag{1.5}$$

$$-\pi\boldsymbol{n} + \nu(\theta)\mathbb{D}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_{NT}, \tag{1.6}$$

$$-\lambda(\theta)\nabla\theta\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_{NT}, \tag{1.7}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{in } \Omega, \tag{1.8}$$

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) \quad \text{in } \Omega. \tag{1.9}$$

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Here we suppose that all functions $\boldsymbol{u}$, θ, π, $\boldsymbol{F}$, ν, λ, $\boldsymbol{u}_0$, θ_0 and θ_D are smooth enough. System (1.1)–(1.9) represents a thermodynamic model for unsteady flows of incompressible heat-conducting Newtonian fluids in exterior-like domain Ω. Modeling of exterior flows past bodies in unbounded domains is not practical from computational point of view. Therefore, the unbounded physical regions are usually truncated to smaller bounded domains by assuming an artificial boundary Γ_N. The Dirichlet boundary conditions (1.4)–(1.5) are prescribed on surfaces of bodies, while the Neumann-type boundary conditions (1.6)–(1.7) are applied on the artificial part Γ_N of the boundary $\partial\Omega$. We refer the reader to [1–6] for discussion and analysis of a number of artificial boundary conditions.

The unknowns in the model are the velocity $\boldsymbol{u}$, temperature θ and pressure π. Throughout the paper, $\boldsymbol{n}$ and $\boldsymbol{\tau}$, respectively, are an outer unit normal and tangential vectors, respectively, to $\partial\Omega$ and $\mathbb{D}(\boldsymbol{u})$ is the so-called rate-deformation tensor field and denotes the symmetric part of $\nabla\boldsymbol{u}$ (the rate of strain tensor) with components

$$D_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Further, data of the problem are as follows: θ_D is a given function representing the distribution of the temperature θ on Γ_{DT}, $\boldsymbol{u}_0$ and θ_0 describe the initial velocity and temperature, respectively, and satisfy the compatibility conditions $\boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{0}$ and $\theta_0(\boldsymbol{x}) = \theta_D(\boldsymbol{x}, 0)$ on Γ_D.

The kinematic viscosity $\nu = \nu(\cdot)$, body force $\boldsymbol{F} = \boldsymbol{F}(\cdot)$ and thermal conductivity $\lambda = \lambda(\cdot)$ are bounded positive continuous functions of temperature. Without any further reference, throughout we assume

$$0 < \nu_1 \le \nu(\xi) \le \nu_2 < +\infty \qquad \forall\xi \in \mathbb{R}\ (\nu_1, \nu_2 = \text{const}), \tag{1.10}$$

$$0 < F_i(\xi) \le C_F < +\infty \qquad \forall\xi \in \mathbb{R}\ (C_F = \text{const}), \tag{1.11}$$

$$0 < \lambda_1 \le \lambda(\xi) \le \lambda_2 < +\infty \qquad \forall\xi \in \mathbb{R}\ (\lambda_1, \lambda_2 = \text{const}). \tag{1.12}$$

Positive constant material coefficients represent the density ϱ and the specific heat c_v, which are in the sequel normalized to 1.

The energy balance equation (1.3) takes into account the phenomena of the viscous energy dissipation, omitted frequently in the Boussinesq model of heat-conducting fluids [7–11]. Eqs. (1.1)–(1.9) represent the system with strong nonlinearities (quadratic growth of $\nabla\boldsymbol{u}$ in the dissipative term $\nu(\theta)\mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{u})$) without appropriate general existence and regularity theory. In [12], Frehse presented an example of discontinuous bounded weak solution $\boldsymbol{U} \in L^\infty \cap H^1$ of nonlinear elliptic system of the type $\Delta\boldsymbol{U} = B(\boldsymbol{U}, \nabla\boldsymbol{U})$, where B is analytic and has quadratic growth in $\nabla\boldsymbol{U}$. However, for scalar problems, such existence and regularity theory is well developed (cf. [13,14]). Nevertheless, the main (open) problem of the system (1.1)–(1.9) consists in the fact that, because of the artificial boundary condition (1.6), we are not able to prove an "a priori" estimate for the convective term $\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})$ in the system of the Navier–Stokes equations. Consequently, we are not able to show that the kinetic energy of the fluid is controlled by the data of the problem and the solutions of (1.1)–(1.9) need not satisfy the energy inequality. This is due to the fact that some uncontrolled "backward flow" can take place at the artificial boundary Γ_N of the truncated domain Ω and one is not able to prove global (in time) existence results by the energy method as in the frequently used case of Dirichlet boundary condition on the whole boundary.

In [15], Kučera and Skalák proved the local-in-time existence and uniqueness of a "weak" solution of the evolution Boussinesq approximations with constant viscosity and thermal conductivity of the heat-conducting incompressible fluids, such that[1]

$$\boldsymbol{u}_t \in L^2(0, T_*; \mathbf{V}^{1,2}_{\sigma,D}), \qquad \boldsymbol{u}_{tt} \in L^2(0, T_*; (\mathbf{V}^{1,2}_{\sigma,D})^*),$$
$$\theta_t \in L^2(0, T_*; V^{1,2}_D), \qquad \theta_{tt} \in L^2(0, T_*; (V^{1,2}_D)^*), \quad 0 < T^* \le T,$$

under some smoothness restrictions on $\boldsymbol{u}_0$, θ_0 and the pressure π. In [16], Beneš and Kučera proved the local weak solvability results for a variational formulation of the appropriate steady problem (but with a constant right-hand side $\boldsymbol{F}$) corresponding to the system (1.1)–(1.9) in a three-dimensional open cylindrical channel. In [17], the author proved the $W^{2,p}$-regularity for steady flows in two-dimensional Lipschitz domains.

Bulíček, Feireisl and Málek [18] considered a complete thermodynamic model for time dependent flows of incompressible homogeneous Newtonian fluids with temperature dependent material coefficients in a fixed bounded three-dimensional domain. The authors established the global weak and the so-called "suitable weak" (in addition, the entropy inequality is required) solutions of the problem with Navier-type boundary conditions for the velocity and Neumann boundary conditions for temperature (zero heat flux across the boundary). The authors extended the results for suitable weak solution, obtained by Leray [19] and Caffarelli, Kohn and Nirenberg [20], from a purely mechanical to a complete thermodynamic model. The analysis of the same model in the spatially periodic setting is presented by Feireisl and Málek [21]. Here the authors replaced the heat flow equation by the balance of total energy that has clear physical background. The resulting system is supplemented with an "entropy inequality" as an extra admissibility condition. The authors proved long-time and large-data existence of a weak solution to the problem describing three-dimensional unsteady flows of incompressible fluids, where the viscosity and heat-conductivity coefficients depend on temperature. In [22], the author proved the

[1] For definitions of corresponding function spaces see Section 2.