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The localized method of approximated particular solutions for near-singular two- and three-dimensional problems

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ABSTRACT

In this paper, the localized method of approximate particular solutions (LMAPS) using radial basis functions (RBFs) has been simplified and applied to near-singular elliptic problems in two- and three-dimensional spaces. The leave-one-out cross validation (LOOCV) is used in LMAPS to search for a good shape parameter of multiquadric RBF. The main advantage of the method is that a small number of neighboring nodes can be chosen for each influence domain in the discretization to achieve high accuracy. This is especially efficient for three-dimension problems. There is no need to apply adaptivity on node distribution near the region containing spikes of the forcing terms. To examine the performance and limitations of the method, we deliberately push the spike of the forcing term to be extremely large and still obtain excellent results. LMAPS is far superior than the compactly supported RBF (Chen et al. 2003) for such elliptic boundary value problems.

1. Introduction

During the past two decades, radial basis functions (RBFs) have became an effective tool for not only surface interpolation and data reconstruction but also for numerical partial differential equations (PDEs). The RBF collocation method was first introduced by Kansa in 1990 [1]. Since then, RBFs have been widely used for solving various kinds of science and engineering problems. Motivated by Kansa's method, several modified versions of RBF collocation methods [2–5] have been developed. Despite the effectiveness and simplicity of these methods, the resultant system of equations is often ill-conditioned, especially when a large amount of collocation points is needed. The selection of the optimal shape parameter of RBFs is also an outstanding research topic. The fundamental issue of the above problems is that the RBFs being used in these methods are globally supported functions. The global RBFs are effective when the solution space of given PDEs is reasonably smooth. On the contrary, when the solution space fluctuates rapidly or presents sharp spikes, the global methods generally do not work well.

Traditional methods such as the finite element method and finite difference method are local methods. They are very effective in dealing with the problems mentioned above. The idea of local collocation methods in the context of RBFs has been introduced by Lee et al. [6] and Tolstykh and Shirobokov [7]. Instead of using all of the interpolation points as in the global case, only data that falls in the influence domain of a given node is used for discretization of the PDE at the node. Based

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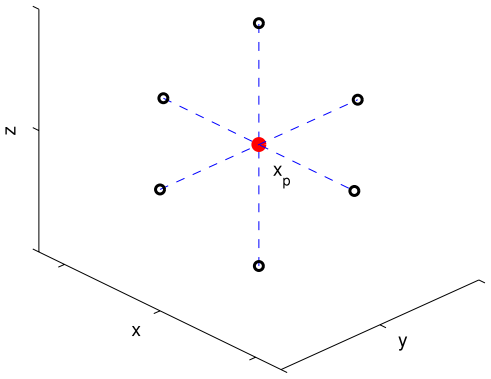

Fig. 1. The 7-point stencil of $\mathbf{x}_p$ in three-dimensional space.

on a similar idea, the global version of the method of approximate particular solutions (MAPS) [2,5] has been extended to the localized version (LMAPS) [8]. LMAPS is very effective for solving large-scale problems. As shown in [8], nearly one million nodes have been used for solving Poisson problems with excellent results. As expected, the selection of a good shape parameter of MQ in LMAPS is much easier than the selection in the global MAPS.

In this paper, we will make improvements to the previous version of LMAPS, and apply it to various near-singular problems in two- or three-dimensional space. A simplified formulation of the original LMAPS is introduced, where the boundary conditions are merged into the governing equations in the discretization process. The leave one out cross validation (LOOCV) [9,10] will be used to select good shape parameter in MQ. Due to the unique feature of the local approach, LMAPS is capable to capture the rapid variation of the solution. Hence, LMAPS is especially attractive for solving near-singular problems as shown in [11]. However, [11] used the smoothing scheme. The enhance nodes near the large spike of the forcing term were needed. The difficulty of solving such near-singular problems becomes more pronounced when the spike of the forcing term gets larger. As indicated in [11], the global methods using RBFs fail to produce acceptable accuracy for the near-singular problems. Chen et al. [11] coupled the method of fundamental solutions with the compactly supported RBFs (CS-RBFs). With the improved version of LMAPS shown in this paper, we are able to solve such near-singular problems without the tedious adaptive effort mentioned in [11], especially we are able to solve three-dimensional problems on extremely irregular domains.

The paper is organized as follows. In Section 2, we briefly extend LMAPS to three-dimensional space. This is a trivial extension due to the radial nature of RBFs. In Section 3, we introduce the simplified formulation of LAMPS. This improved the efficiency of the method. The particular solution of the MQ RBF for Laplace operator in 2D and 3D are given. In Section 4, we also briefly explain LOOCV for selecting a good shape parameter in MQ RBFs. In Section 5, to demonstrate the effectiveness of LMAPS for solving near-singular problems, we tested our methods on the following PDEs: two Poisson's equations on unit cube or irregular domains in 2D and 3D; and a modified Helmholtz problem on Stanford Bunny domain in 3D. In Section 6, we make some comments and draw conclusions.

2. LMAPS

In this section, we give a brief review of LMAPS in three-dimensional space. Let L be a linear second-order elliptic partial differential operator and B be a boundary operator. We consider the following boundary value problem

$$Lu(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1}$$

$$Bu(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ is a bounded and closed two- or three-dimensional domain with a boundary $\partial\Omega$. We assume that the above boundary value problem has a unique solution for sufficiently smooth nonhomogeneous term f and boundary data g.

Let $\{\mathbf{x}_j\}_{j=1}^{n}$ be the interpolation points inside the domain Ω. In this paper, the interpolation points are distributed uniformly in the domain. For any point $\mathbf{x}_p \in \Omega$, we create an influence domain Ω_p, which contains a region formed by the n_s nearest neighboring interpolation points $\{\mathbf{x}_j\}_{j=1}^{n_s}$ to $\mathbf{x}_p$ including $\mathbf{x}_p$. We usually choose five nearest neighboring points for each node in 2D. For three-dimensional problems, we intend to select more neighboring points, about 7–25 points, than that in two-dimensional domain for higher dimensional space. Figs. 1 and 2 show the seven and nineteen nearest neighboring interpolation points of $\mathbf{x}_p$, respectively, in 3D.

By the method of particular solutions (MAPS) [2], $u(\mathbf{x}_p)$ can be approximated by a linear combination of n_s RBFs in the following form:

$$u(\mathbf{x}_p) \simeq \hat{u}(\mathbf{x}_p) = \sum_{j=1}^{n_s} a_j \Phi\left(\left\|\mathbf{x}_p - \mathbf{x}_j\right\|\right), \tag{3}$$