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A saddle point least squares approach to mixed methods

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ABSTRACT

We investigate new PDE discretization approaches for solving variational formulations with different types of trial and test spaces. The general mixed formulation we consider assumes a stability LBB condition and a data compatibility condition at the continuous level. We expand on the Bramble–Pasciak's least square formulation for solving such problems by providing new ways to choose approximation spaces and new iterative processes to solve the discrete formulations. Our proposed method has the advantage that a discrete inf − sup condition is automatically satisfied by natural choices of test spaces (first) and corresponding trial spaces (second). In addition, for the proposed iterative solver, a nodal basis for the trial space is not required. Applications of the new approach include discretization of first order systems of PDEs, such as div − curl systems and time-harmonic Maxwell equations.

1. Introduction

Over the last two decades, there have been many advances in applying finite element least squares methods to approximate first order systems of PDEs, [1–11]. However, when compared to the more established field of finite element methods for elliptic problems, a unified theoretical framework for least squares approximation of solutions of first order systems of PDEs is missing. Our proposed framework provides powerful preconditioning techniques, has efficient error estimators, is suitable to multilevel techniques, and leads to robust and easy to implement solvers. We combine known theory and discretization techniques for approximating elliptic problems and for symmetric saddle point problems (see [12–23]) to obtain a unified framework for discretizing variational formulations with different types of test and trial spaces. In particular, the framework can be applied to least square approximation for a large class of first order systems of PDEs.

For the applications we consider, the solution spaces are L^2 type spaces, and the data can reside in weak negative norm spaces. We require that the test spaces be H^1 type spaces with suitable boundary conditions, and the discrete test spaces be conforming finite element spaces built using the action of the continuous differential operator associated with a given problem. Among the advantages of the method are the following: the discretization leads to saddle point variational formulation with automatic discrete inf–sup condition, and assembly of stiffness matrices for the trial spaces is avoided.

The general abstract problem that we plan to discretize using a *Saddle Point Least Squares Method* (SPLS) is: Find $p \in Q$ such that

$$b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \text{for all } \mathbf{v} \in \mathbf{V} \quad \text{or} \quad B^* p = \mathbf{f}, \tag{1.1}$$

where $\mathbf{V}$ and Q are infinite dimensional Hilbert spaces and $b(\cdot, \cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, that satisfies a standard inf–sup condition, and $\mathbf{f} \in \mathbf{V}^*$ belongs to the range of B^*. In the special case when the operator B associated with

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the form $b(\cdot,\cdot)$ is injective, our method can be viewed as a conforming Petrov–Galerkin method. From the point of view of choosing the discrete spaces, our method can be characterized as the dual of the Discontinuous Petrov–Galerkin (DPG) method introduced by Demkowicz and Gopalakrishnan in [24,25]. While both methods have strong connections with the least squares and minimum residual techniques, the proposed discretization process *stands apart* from the DPG approach because of the *opposite order* and *different ways* in which the trial and test spaces are chosen.

We propose the following main steps of our *saddle point least square discretization* method:

Step 1 Reduce the general problem (1.1) to a *saddle point least square* formulation, using the natural inner product $a_0(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$ as the $(1, 1)$ form of the saddle point system (see problem (2.12)).

Step 2 Choose a standard *conforming approximation space* $\mathbf{V}_h$ for the *variational space* $\mathbf{V}$.

Step 3 Construct a discrete *trial space* $\mathcal{M}_h \subset Q$ using the operator B associated with the form $b(\cdot,\cdot)$. For example, take $\mathcal{M}_h := \mathcal{C}^{-1}B\mathbf{V}_h$, or $\mathcal{M}_h := \tilde{\mathcal{Q}}_h\mathcal{C}^{-1}B\mathbf{V}_h$, where $\mathcal{C}^{-1}$ is the Riesz representation operator for the space Q and $\tilde{\mathcal{Q}}_h$ is a *projection* from Q to a subspace $\tilde{\mathcal{M}}_h$. The pair $(\mathbf{V}_h, \mathcal{M}_h)$ will automatically satisfy a discrete inf–sup condition.

Step 4 Write the discrete version of SPLS formulation on $(\mathbf{V}_h, \mathcal{M}_h)$, see (3.3), and replace $a_0(\cdot,\cdot)$ by an equivalent form $a_{prec}(\cdot,\cdot)$ on $\mathbf{V}_h\times\mathbf{V}_h$.

Step 5 Solve the new discrete SPLS problem using an Uzawa type iterative process that requires only the action of the preconditioner associated with $a_{prec}(\cdot,\cdot)$ and the action of $\mathcal{C}^{-1}B$ or $\tilde{\mathcal{Q}}_h C^{-1}B$.

The paper is organized as follows. In Section 2, we introduce notation and review basic abstract results needed to describe the method. We also include here the first step of reduction of a mixed problem to a Saddle Point Problem (SPP). In Section 3, we present the discretization part of the method (Step 2 and Step 3) and discuss the choice of discrete spaces and their approximability. Uzawa type Iterative solvers without trial space bases are presented in Section 4. The special choice of spaces with discrete inf–sup condition and their approximability properties are presented in Section 5. In Section 6 we present an example of SPLS discretization for a div–curl system. The Appendix contains some important functional analysis results needed for the proofs of the paper.

2. Notation and background

In this section, we start with a review of the notation of the classical SPP theory and introduce the spaces, the operators and the norms for the general abstract case. We let $\mathbf{V}$ and Q be two Hilbert spaces with inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, with the corresponding induced norms $|\cdot|_{\mathbf{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$. The dual pairings on $\mathbf{V}^*\times\mathbf{V}$ and $Q^*\times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $\mathbf{V}^*$ and Q^* denote the duals of $\mathbf{V}$ and Q, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate the operators
$\mathcal{A} : V \to V^*$ and $\mathcal{C} : Q \to Q^*$ defined by

$$\langle\mathcal{A}\mathbf{u},\mathbf{v}\rangle = a_0(\mathbf{u},\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v}\in\mathbf{V}$$

and

$$\langle\mathcal{C}p, q\rangle = (p, q) \quad \text{for all } p, q \in Q.$$

The operators $\mathcal{A}^{-1} : V^* \to V$ and $\mathcal{C}^{-1} : Q^* \to Q$ are called the Riesz-canonical isometries and satisfy the following properties

$$a_0(\mathcal{A}^{-1}\mathbf{u}^*,\mathbf{v}) = \langle\mathbf{u}^*,\mathbf{v}\rangle, \qquad |\mathcal{A}^{-1}\mathbf{u}^*|_{\mathbf{V}} = \|\mathbf{u}^*\|_{\mathbf{V}^*}, \quad \mathbf{u}^*\in\mathbf{V}^*,\ \mathbf{v}\in\mathbf{V}, \tag{2.1}$$

$$(\mathcal{C}^{-1}p^*, q) = \langle p^*, q\rangle, \qquad \|\mathcal{C}^{-1}p^*\| = \|p^*\|_{Q^*}, \quad p^*\in Q^*,\ q\in Q. \tag{2.2}$$

Next, we suppose that $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V}\times Q$, satisfying the inf–sup condition. More precisely, we assume that

$$\inf_{p\in Q}\sup_{\mathbf{v}\in\mathbf{V}}\frac{b(\mathbf{v},p)}{\|p\|\,|\mathbf{v}|} = m > 0 \tag{2.3}$$

and

$$\sup_{p\in Q}\sup_{\mathbf{v}\in\mathbf{V}}\frac{b(\mathbf{v},p)}{\|p\|\,|\mathbf{v}|} = M < \infty. \tag{2.4}$$

Here, and throughout this paper, the "inf" and "sup" are taken over nonzero vectors. With the form b, we associate the linear operators $B : V \to Q^*$ and $B^* : Q \to V^*$ defined by

$$\langle B\mathbf{v}, q\rangle = b(\mathbf{v}, q) = \langle B^*q,\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{V},\ q\in Q.$$

Let $\mathbf{V}_0$ be the kernel of B or $\mathcal{C}^{-1}B$, i.e.,

$$\mathbf{V}_0 = Ker(B) = \{\mathbf{v}\in\mathbf{V}|\ B\mathbf{v} = 0\} = \{\mathbf{v}\in\mathbf{V}|\ \mathcal{C}^{-1}B\mathbf{v} = 0\}.$$

Due to (2.4), $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$.