Computers and Mathematics with Applications 70 (2015) 2933–2945

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Analysis of a splitting–differentiation population model leading to cross-diffusion

Gonzalo Galiano *, Virginia Selgas

Department of Mathematics, Universidad de Oviedo, c/ Calvo Sotelo, 33007 Oviedo, Spain

ARTICLE INFO

Article history:
Received 12 July 2015
Accepted 12 October 2015
Available online 1 November 2015

Keywords:
Cross-diffusion system
Population dynamics
Contact inhibition
Splitting and differentiation
Numerical simulations
Existence of solutions

ABSTRACT

Starting from the dynamical system model capturing the splitting–differentiation process of populations, we extend this notion to show how the speciation mechanism from a single species leads to the consideration of several well known evolution cross-diffusion partial differential equations.

Among the different alternatives for the diffusion terms, we study the model introduced by Busenberg and Travis, for which we prove the existence of solutions in the one-dimensional spatial case. Using a direct parabolic regularization technique, we show that the problem is well posed in the space of bounded variation functions, and demonstrate with a simple example that this is the best regularity expected for solutions.

We numerically compare our approach to other alternative regularizations previously introduced in the literature, for the particular case of the contact inhibition problem. Simulation experiments indicate that the numerical scheme arising from the approximation introduced in this article outperforms those of the existent models from the stability point of view.

1. Mathematical model and main result

1.1. The splitting–differentiation model in terms of ODEs

In [1], Sánchez-Palencia analyzes the situation in which a species, with population density U, splits, due to a number of factors, into two different species with population densities U_1 and U_2, but still keeping the original ecological behavior.

Thus, we assume that U followed a logistics law until time t^*, i.e.

$$U'(t) = U(t)\big(\alpha - \beta U(t)\big), \qquad \text{for } t \in (0, t^*), \quad U(0) = U_0 > 0. \tag{1}$$

Then, after splitting, (U_1, U_2) is assumed to satisfy the Lotka–Volterra system

$$U_i'(t) = U_i(t)\big(\alpha - \beta\big(U_1(t) + U_2(t)\big)\big), \quad \text{for } t \in (t^*, T), \qquad U_i(t^*) = U_{i0} > 0, \tag{2}$$

for $i = 1, 2$, with $U_{10} + U_{20} = U(t^*)$. Note that, under this splitting *without differentiation*, $U_1 + U_2$ still satisfies (1) for $t \geq t^*$.

Although problem (1) has a unique non-trivial equilibrium, $U_\infty = \alpha/\beta$, problem (2) has a continuum set of non-trivial equilibria given by all the combinations of $U_{1\infty} \geq 0$ and $U_{2\infty} \geq 0$ such that $U_{1\infty} + U_{2\infty} = \alpha/\beta$.

* Corresponding author. Tel.: +34 985103343; fax: +34 985103354.
E-mail addresses: galiano@uniovi.es (G. Galiano), selgasvirginia@uniovi.es (V. Selgas).

http://dx.doi.org/10.1016/j.camwa.2015.10.005
0898-1221/

In [1], the author analyzes how the differentiation of populations U_1, U_2 after splitting, understood as a perturbation in the Lotka–Volterra coefficients, affects the equilibrium of the system. By splitting *with differentiation* we mean that (U_1, U_2) is a solution of the following problem:

$$U_i'(t) = U_i(t)\big(\alpha_i - \big(\beta_{i1}U_1(t) + \beta_{i2}U_2(t)\big)\big), \quad \text{for } t \in (t^*, T), \qquad U_i(t^*) = U_{i0} > 0, \tag{3}$$

for $i = 1, 2$, with $U_{10} + U_{20} = U(t^*)$. Observe that, under this splitting, $U_1 + U_2$ does not satisfy, in general, problem (1) for $t \geq t^*$.

The main conclusion of [1] is that the differentiation mechanism selects, in general, a unique solution for the equilibrium system, having therefore a stabilizing effect.

1.2. The splitting model in terms of PDEs: cross-diffusion

In this article, we extend the previous dynamical system models to the case of space dependent population densities. We start considering the dynamics of one single species population satisfying

$$\begin{cases} \partial_t u - \operatorname{div} J(u) = f(u) & \text{in } Q_{(0,t^*)}, \\ J(u) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_{(0,t^*)}, \\ u(0, \cdot) = u_0 \geq 0 & \text{on } \Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary, $\partial\Omega$, $Q_{(0,t^*)} = (0, t^*) \times \Omega$, and $\Gamma_{(0,t^*)} = (0, t^*) \times \partial\Omega$ is the parabolic boundary of $Q_{(0,t^*)}$. The vector $\boldsymbol{\nu}$ is the outward canonical normal to $\partial\Omega$. The growth-competition term is assumed to have the logistic form $f(u) = u(\alpha - \beta u)$, and the flow to be given by $J(u) = u\nabla u + u\mathbf{q}$.

As it is well known, the term $u\nabla u$ captures the individuals aversion to overcrowding, while $\mathbf{q}$ is usually determined by an environmental potential, $\mathbf{q} = -\nabla\varphi$, whose minima represent attracting points for the populations. For biological background and origins of the model see, for instance, [2].

After splitting, the new two populations, u_1 and u_2, satisfy, for $i = 1, 2$,

$$\begin{cases} \partial_t u_i - \operatorname{div} J_i(u_1, u_2) = f_i(u_1, u_2) & \text{in } Q_{(t^*,T)}, \\ J_i(u_1, u_2) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_{(t^*,T)}, \\ u_i(t^*, \cdot) = u_{i0} & \text{on } \Omega, \end{cases} \tag{5}$$

with u_{i0} such that $u_{10} + u_{20} = u(t^*, \cdot)$, and with J_i and f_i to be defined.

Assuming, like in the dynamical system model, that the possible differentiation process only takes place through the growth and the inter- and intra-competitive behavior of the new species implies that the split flows must satisfy $J_1(u_1, u_2) + J_2(u_1, u_2) = J(u_1 + u_2)$, that is

$$J_1(u_1, u_2) + J_2(u_1, u_2) = (u_1 + u_2)\nabla(u_1 + u_2) + (u_1 + u_2)\mathbf{q}.$$

Being clear the way of defining the linear transport term of J_i, the nonlinear diffusive term admits several reasonable decompositions. For instance, in [3], the following splitting was considered

$$J_i(u_1, u_2) = u_i\nabla u_i + b_i\nabla(u_1u_2) + u_i\mathbf{q}, \tag{6}$$

with $b_i \geq 0$, and $b_1 + b_2 = 1$. Under this splitting, problem (5) takes the form of the cross-diffusion model introduced by Shigesada et al. [4], for which a thorough mathematical analysis does exist, see for instance [5–10] for numerical approaches, [11–15] for analytical and qualitative results, or [16–18] for applications.

In this paper, we consider the alternative splitting

$$J_i(u_1, u_2) = u_i\nabla(u_1 + u_2) + u_i\mathbf{q}, \tag{7}$$

which brings problem (5) to the form of the Busenberg and Travis model [19]. Although apparently simpler than (6), no general proof of existence of solutions does exist for problem (5) with flows given by (7). However, some partial results related to the cell-growth contact-inhibition problem may be found in [20–23], as well as in [24,25] for other specific situations.

1.3. Differentiation after splitting

According to the species behavior after splitting, we consider two problems arising from two different sets of Lotka–Volterra terms:

$$f_i(u_1, u_2) = u_i(\alpha - \beta(u_1 + u_2)), \qquad \text{(non-differentiation)} \tag{8}$$

$$f_i(u_1, u_2) = u_i\big(\alpha_i - \big(\beta_{i1}u_1 + \beta_{i2}u_2\big)\big), \qquad \text{(differentiation)} \tag{9}$$

for $i = 1, 2$. We shall refer to problem (5) with flows given by (7) and with f_i given by (8) and (9) as to **problems (ND) and (D)**, respectively. Observe that these are the PDE versions that generalize the non-differentiation and differentiation ODE problems (2) and (3), introduced in [1].