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A hybridized iterative algorithm of the BiCORSTAB and GPBiCOR methods for solving non-Hermitian linear systems

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ABSTRACT

In this study, we derive a new iterative algorithm (including its preconditioned version) which is a hybridized variant of the biconjugate A-orthogonal residual stabilized (BiCORSTAB) method and the generalized product-type solvers based on BiCOR (GPBiCOR) method. The proposed method, which is named GPBiCOR(m, ℓ) similarly to the GPBiCG(m, ℓ) method proposed by Fujino (2002), can be regarded as an extension of the BiCORSTAB2 method introduced by Zhao and Huang (2013). Inspired by Fujino's idea for improving the BiCGSTAB2 method, in the established GPBiCOR(m, ℓ) method the parameters computed by the BiCORSTAB method are chosen at successive m iteration steps, and afterwards the parameters of the GPBiCOR method are utilized in the subsequent ℓ iteration steps. Therefore, the proposed method can inherit the low computational cost of BiCORSTAB and the attractive convergence of GPBiCOR. Extensive numerical convergence results on selected real and complex matrices are shown to assess the performances of the proposed GPBiCOR(m, ℓ) method, also against other popular non-Hermitian Krylov subspace methods.

1. Introduction

In many computational science and engineering applications, there is a strong need of fast and efficient solutions of large sparse linear systems

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where A is an $n \times n$ non-Hermitian and possibly indefinite matrix, and $\boldsymbol{b}$ is the right-hand side vector of length n. There are two classes of numerical methods for solving the linear system (1) on modern computers, namely direct and iterative solvers. Sparse direct methods are generally very accurate, robust, and predictable in terms of both storage and algorithmic cost (e.g. see [1]). However, they tend to be very expensive for solving large problems. A large number of researchers are still interested in developing robust iterative solvers, especially the well-known class of Krylov subspace methods, as they only require matrix–vector products; refer to [2–5]. As we already know, if the coefficient matrix A is Hermitian positive definite, the conjugate gradient (CG) method [6] is a very good choice. In addition, if the coefficient matrix A is complex symmetric,

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i.e. $A = A^T$ and $A \neq A^H$, Krylov subspace methods can exploit symmetry during the iterative process; refer, e.g., to our recent work [7,8] about the SCBiCG class of iterative algorithms. In the case, the coefficient matrix A is substantially non-Hermitian (e.g., see [2,3] and references therein), two popular non-Hermitian iterative solvers are the generalized minimal residual (GMRES) method proposed by Saad and Schultz [9], and the generalized conjugate residual (GCR) method proposed by Eisenstat, Elman and Schultz [10]. Both the GMRES and GCR methods search the *optimal* approximation $\boldsymbol{x}_m$ for which the 2-norm of the residual $\boldsymbol{r}_m$ is minimal over the Krylov space $\mathcal{K}_m(A, \boldsymbol{r}_0) = \text{span}\{\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{m-1}\boldsymbol{r}_0\}$ for $m = 1, 2, \ldots$, where $\boldsymbol{r}_0$ denotes the initial residual. However, due to the linearly increasing memory and computational costs with the iteration count, the two algorithms are often restarted after each cycle of, say, m iterations. The restarted GMRES and GCR variants, dubbed by GMRES(m) [2,9, pp. 179–180] and GCR(m) [10], however, do not share the optimality property of the original methods, and in practice they show some difficulties to converge on difficult problems. Some strategies to recover the optimal convergence behavior, while keeping the cost per iteration roughly constant, such as deflation and augmentation accelerating techniques, are described in [4,11–15].

On the other side, a prominent *non-optimal* Krylov subspace method for non-Hermitian systems is the biconjugate gradient (BiCG) method proposed in [16,17]. However, the BiCG method displays irregular convergence behavior in many practical applications and requires two matrix–vector products, one with A and one with the transpose conjugate matrix A^H per iteration step. Several variants of the BiCG method have been proposed to enhance its overall performance, such as the conjugate gradient squared (CGS) method proposed by Sonneveld [18], the generalized CGS (GCGS) method proposed by Fokkema, Sleijpen and van der Vorst [19], Freund and Nachtigal's transpose-free quasi-minimal residual (TFQMR) method [20], van der Vorst's biconjugate gradient stabilized (BiCGSTAB) method [21], the BiCGSTAB2 method by Gutknecht [22], the BiCGSTAB(ℓ) method by Sleijpen and Fokkema [23], and the generalized product-type method based on BiCG (GPBiCG) method introduced by Zhang [24,25]. Moreover, the reborn induced dimension reduction (abbreviated as IDR(s)) method, which is also an efficient short-recurrence family of Krylov subspace methods, was proposed by Sonneveld and van Gijzen in [26]. A different approach to generalize BiCGSTAB was proposed by Yeung and Chan, whose ML(k)BiCGSTAB method [27] is a BiCGSTAB variant based on multiple left Lanczos starting vectors; for a comparative study of the IDR(s) and ML(k)BiCGSTAB methods, see [28].

It is a question to determine the classes of problems for which one algorithm is more numerically stable and memory efficient than others. For example, in many problems the CGS method is significantly faster than the BiCG method, but the convergence behavior is much more irregular, and this can potentially affect the final convergence rate and accuracy of the solution. Instead of squaring the BiCG polynomial [16] as in CGS, the authors of [19] consider products of two nearby BiCG polynomials, which results in the GCGS method. The BiCGSTAB2 and BiCGSTAB(ℓ) methods integrate both first and second (or higher) degree auxiliary polynomials, amending the convergent stability of the BiCGSTAB method on many problems. In contrast, the GPBiCG method utilizes only second degree auxiliary polynomials. Therefore each GPBiCG iteration step is slightly more expensive than other product-type iterative methods; similarly, refer to Table 1 about the cost for computing the scalar parameters. However, the idea behind the GPBiCG method implies that it is further possible to develop hybridized variants which attempt to select the different parameters of the algorithm to minimize its final cost; see [24] for details.

In recent years, we [29,30] proposed a novel class of efficient *non-optimal* iterative methods for solving non-Hermitian linear systems (1), dubbed biconjugate A-orthogonal residual (BiCOR) family. The BiCOR family of solvers requires only $\mathcal{O}(n)$ extra storage in addition to the matrix and performs $\mathcal{O}(n)$ operations per iteration, often shows competitive and sometimes superior performance against other popular Krylov subspace solvers in solving different scientific and engineering applications [31,32]. The first variant, named the BiCOR method, can have much smoother and faster convergence behavior than the BiCG method. The same residual polynomial of the BiCOR method is utilized and squared in the second variant, the CORS method [29]. The convergence of the CORS method may be typically more irregular than that of the BiCOR method when the BiCOR method has a jagged convergence curve. In order to overcome this convergence problem, the BiCORSTAB method has been established exploiting the same philosophy behind the BiCGSTAB method. While the BiCORSTAB method works well in a lot of cases, it still has quite erratic oscillation in some difficult problems with complex spectrum; see e.g., the experiments reported in [29,32] on the representative matrix problems. By applying the ideas behind the BiCORSTAB, CORS, CGS and BiCGSTAB methods, a few generalized methods have been recently proposed such as the generalized product-type solvers based on BiCOR (GPBiCOR) method introduced by Zhao, Huang, Jing et al. [33], the generalized CORS (GCORS) method by Zhang and Dai [34], the QMOR method also by Zhang and Dai [35], the TFQMR-like variant of the CORS (TFQMORS) method by Zhang and Dai [36], and the BiCORSTAB2 method by Zhao and Huang [37]. Again, the GPBiCOR method utilizes always the second degree polynomial only similar to the case of GPBiCG. Therefore, cost of the GPBiCOR method per iteration step is slightly more expensive than those of other product-type iterative methods such as CORS and BiCORSTAB. By the way, it is also noted that almost simultaneously the authors of [38] independently constructed the hybrid BiCR methods for solving nonsymmetric linear systems via replacing the BiCG part in the residual polynomial of the hybrid BiCG variants with BiCR [39], but the comparison of the BiCOR family of solvers [29,30] and the hybrid BiCR variants [38] is not the emphasis of this present study.

A mechanism of the GPBiCOR method has indicated further possibility of constructing a new hybridized variant in which one can vary from the CORS and BiCORSTAB methods to the BiCORSTAB2 method with suitable choice of parameters [33]. In this paper, we investigate this possibility. The research was prompted by the pioneering work about the GPBiCG(m, ℓ) method [40], which makes an attempt to construct a hybridized iterative procedure for reducing the cost of the GPBiCG iterations. The GPBiCOR method is considerably more competitive than the GPBiCG method, see [33] for details, but this