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Period two implies chaos for a class of multivalued maps: A naive approach

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ABSTRACT

On the background of our earlier results concerning the coexistence of infinitely many periodic orbits, we present a new theorem dealing with a large class of one-dimensional multivalued maps with monotone margins and connected values. If a nontrivial ($n > 1$) n-orbit occurs then, according to our theorem, these maps possess a single-valued chaotic selection, on a compact subinterval.

1. Introduction

The main aim of this paper is to demonstrate that there exists a large class of one-dimensional multivalued maps exhibiting chaos. These maps have a very strong forcing property. More concretely, if there exists a positive integer $n > 1$ such that they admit an n-orbit, then the coexistence of k-orbits occurs, for each $k \in \mathbb{N}$. At the same time, there also exists a single-valued continuous selection on a compact interval which is chaotic, practically in an arbitrary way. Roughly speaking, "*period two implies chaos*" here (whence the title). In fact, we will show that "*any nontrivial period is equivalent with many sorts of chaos*", but this requires to be explained in more detail.

In order to understand these phenomena in a deeper way, let us start with a survey of similar results for single-valued continuous functions. There are two classical results of this type, namely the Sharkovsky cycle coexistence theorem in [1] and the one due to Li and Yorke saying that "*period three implies chaos*" in [2]. The Sharkovsky theorem is based on a new (Sharkovsky) ordering of positive integers:
$3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd 2\cdot 9 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd 2^2\cdot 9 \rhd \cdots 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd 2^n\cdot 9 \rhd \cdots \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1.$

Theorem 1 (*Sharkovsky' 64*). *Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If there is some $a \in \mathbb{R}$ which forms a primary n-orbit $\mathcal{O}$ of f, i.e.*

$$\mathcal{O} = \underbrace{\{a, f(a), \ldots, f^{n-1}(a)\}}_{\text{mutually different elements}}, \quad \text{where } f^n(a) = a,$$

then for each $k \lhd n$ (in the Sharkovsky ordering) there is, for some $b \in \mathbb{R}$, also a primary k-orbit of f.

Remark 1. The special case of Theorem 1, for $n = 3$, was obtained much later, but independently, in [2]. This forcing property is in principle one-dimensional, because the analogous criteria in $\mathbb{R}^n$ are very drastic, for $n > 1$ (cf. [3]).

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Theorem 2 (*Li and Yorke, 1975*)**.** *Let a continuous function $f: \mathbb{R} \to \mathbb{R}$ admit a primary 3-orbit. Then there exists an uncountable subset $S \subset \mathbb{R}$ (called a scrambled set) with the following properties:*
(i) $\limsup_{m\to\infty} |f^m(x) - f^m(y)| > 0$, *for all* $x, y \in S, x \neq y$,
(ii) $\liminf_{m\to\infty} |f^m(x) - f^m(y)| = 0$, *for all* $x, y \in S, x \neq y$.

Remark 2. The properties (i), (ii) holding on a scrambled set $S \subset \mathbb{R}$ are nowadays called *chaos in the sense of Li–Yorke*.

Theorem 2 was extended in [4,5] to "*period* $\neq 2^n$ *implies chaos*". In [6], this implication was replaced by the equivalence, but the chaos had this time a different meaning, namely that f has a *positive topological entropy*, i.e. $h(f) > 0$. For its definition, see e.g. [7]. This is still equivalent with the existence of a *horseshoe* of f^k, for some $k \geq 1$, i.e. with the existence of an interval $I \subset \mathbb{R}$ and disjoint open subintervals K_1 and K_2 of I such that $f^k(K_1) = f^k(K_2) = I$, for some $k \geq 1$, (cf. [7]) as well as with the *topological transitivity* of f, i.e. with the existence of nonempty open subsets U and V of $\mathbb{R}$ such that $f^k(U) \cap V \neq 0$, for some $k \geq 1$. In fact, all the above equivalent properties coincide with the *chaos in the sense of Devaney* (see e.g. [8]), because the sole transitivity implies, on intervals in $\mathbb{R}$, that the set of periodic points of f is dense and that f has a sensitive dependence on initial conditions. Moreover, "*transitivity implies period 6*" (cf. [9,10]). For the related definitions and more details, see e.g. [11].

On the other hand, neither Theorem 1 nor Theorem 2 can be applied, via Poincaré's translation operators, to scalar ordinary differential equations. To be more precise, let us consider the scalar equation

$$x' = f(t, x), \tag{1}$$

where (for the sake of simplicity) $f \in C(\mathbb{R}^2, \mathbb{R})$ satisfies the linear growth restrictions

$$|f(t, x)| \leq \alpha + \beta|x|, \quad \text{for all } (t, x) \in \mathbb{R}^2,$$

and assume that

$$f(t, x) \equiv f(t + 1, x).$$

In view of Theorems 1 and 2, assume for a moment the unique solvability of (1). It is well-known (see e.g. [12, Theorem 9.1] that every bounded solution of (1) on the half-line is either 1-periodic or asymptotically 1-periodic which excludes the existence of subharmonic k-periodic solutions, for any $k > 1$. Another, even more transparent argument, concerns just the associated *Poincaré translation operators* $T_n: \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, along the trajectories of (1), i.e.

$$T_n(x_0) := \{x(n, x_0): x(\cdot, x_0) \text{ is a solution of (1), where } x(0, x_0) = x_0\}. \tag{2}$$

Because of a unique solvability of (1), solutions of (1) depend continuously on initial conditions by which T_n becomes completely continuous. Moreover, T_n is still strictly increasing (otherwise, a contradiction with uniqueness), and subsequently also homeomorphic. The monotonicity of T_n again excludes the existence of k-periodic points of T_n which determine in a one-to-one way k-periodic solutions of (1), for any $k > 1$.

Since, in the lack of uniqueness, the Poincaré operators are obviously multivalued, this was for us a stimulation to consider versions of Theorems 1 and 2 for multivalued maps, possibly applicable to differential equations and inclusions. Let us note that, according to the result of Orlicz, ordinary differential equations are generically uniquely solvable.

2. Sharkovsky-type theorems for multivalued maps

Hence, consider now multivalued maps $\varphi: \mathbb{R} \multimap \mathbb{R}$ (i.e. $\varphi: \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$).

Definition 1. By an *orbit of kth-order* (*k-orbit*) to a multivalued map $\varphi: \mathbb{R} \multimap \mathbb{R}$, we mean a sequence $\{x_i\}_{i=0}^{k-1}$ such that
(i) $x_{i+1} \in \varphi(x_i)$, $i = 0, 1, \ldots, k-2$,
(ii) $x_0 \in \varphi(x_{k-1})$,
(iii) this orbit is not a product orbit formed by going p-times around a shorter orbit of mth-order, where $mp = k$.

If still

(iv) $x_i \neq x_j$, for $i \neq j$; $i, j = 0, 1, \ldots, k-1$, then we speak about a *primary orbit of kth-order* (*primary k-orbit*).

Definition 2. A multivalued map $\varphi: \mathbb{R} \multimap \mathbb{R}$ is *upper semicontinuous* (u.s.c.) if, for any open $U \subset \mathbb{R}$, the small preimage $\{x \in \mathbb{R}: \varphi(x) \subset U\}$ of φ is also open.

For the Poincaré operators T_n in (2), their upper semicontinuity (which is always the case) is equivalent to the closedness of their graph Γ_{T_n}, where

$$\Gamma_{T_n} := \{(x, y) \in \mathbb{R}^2: x \in \mathbb{R},\ y \in T_n(x)\}.$$

Moreover, for every $x \in \mathbb{R}$, the set of values $\{T_n(x)\}$ of T_n consists either of a singleton or of a compact interval and $T_n^m = T_{nm}$, $m \in \mathbb{N}$. For more details, (see e.g. [13, Chapter III.4]).

For these maps, we were also able to prove the following theorem.