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The Euler–Maruyama approximation for the asset price in the mean-reverting-theta stochastic volatility model

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ABSTRACT

Stochastic differential equations (SDEs) have been used to model an asset price and its volatility in finance. Lewis (2000) [10] developed the mean-reverting-theta processes which can not only model the volatility but also the asset price. In this paper, we will consider the following mean-reverting-theta stochastic volatility model

$$\begin{aligned} dX(t) &= \alpha_1(\mu_1 - X(t))dt + \sigma_1\sqrt{V(t)}X(t)^\theta dw_1(t), \\ dV(t) &= \alpha_2(\mu_2 - V(t))dt + \sigma_2 V(t)^\beta dw_2(t). \end{aligned}$$

We will first develop a technique to prove the non-negativity of solutions to the model. We will then show that the EM numerical solutions will converge to the true solution in probability. We will also show that the EM solutions can be used to compute some financial quantities related to the SDE model including the option value, for example.

1. Introduction

In general, the rate of the change of an asset price $X(t)$ consists of random changes and deterministic changes. The well-known Black–Scholes [1] model of the asset price is described by the linear SDE

$$dX(t) = \alpha_1 X(t)dt + \sigma_1 X(t)dw_1(t), \tag{1.1}$$

where w_1 is a scalar Brownian motion and the rate of return α_1 and the volatility σ_1 are assumed to be constants. Later, Vasicek [2] developed the mean-reverting model and Cox, Ingersoll and Ross (CIR) [3] modified it into the mean-reverting square root process which has the SDE form

$$dX(t) = \alpha_1(\mu_1 - X(t))dt + \sigma_1\sqrt{X(t)}dw_1(t). \tag{1.2}$$

This SDE has been widely used to model the interest rates and volatility (see also [4,5]). Moreover, according to the empirical studies, many authors have shown that the volatility is a stochastic process and it can be modelled by an SDE in many situations (see e.g. [6,3,7–9]). In particular, Hull and White [8] observed that the instantaneous variance $V = \sigma_1^2$ is governed by another Brownian motion w_2 and can be described by the SDE

$$dV(t) = \alpha_2 V(t)dt + \sigma_2 V(t)dw_2(t), \tag{1.3}$$

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where α_2, σ_2 are constants. Heston [7] proposed to model the variance by the mean reverting square root process

$$dV(t) = \alpha_2(\mu_2 - V(t))dt + \sigma_2\sqrt{V(t)}dw_2(t). \tag{1.4}$$

Lewis [10] developed this into the more general mean-reverting-theta process

$$dV(t) = \alpha_2(\mu_2 - V(t))dt + \sigma_2 V(t)^\theta dw_2(t), \tag{1.5}$$

which can not only model the volatility but also the asset price (see also [11,12]), where $\theta \geq 1/2$.

Accordingly, we will, in this paper, consider the following mean-reverting-theta stochastic volatility model

$$\begin{aligned} dX(t) &= \alpha_1(\mu_1 - X(t))dt + \sigma_1\sqrt{V(t)}X(t)^\theta dw_1(t), \\ dV(t) &= \alpha_2(\mu_2 - V(t))dt + \sigma_2 V(t)^\beta dw_2(t). \end{aligned} \tag{1.6}$$

This SDE model has no explicit solutions. Hence numerical techniques have become one of the most popular and powerful tools to find the approximate solution (see [13–18]). In the case when $1/2 \leq \beta, \theta \leq 1$,the strong convergence (in L^2) of the Euler–Maruyama (EM) approximate solution has been established by Mao et al. [19]. On the other hand, some empirical studies show that the most successful continuous-time models of the short-term rate in capturing the dynamics are those that allow the volatility of interest changes to be highly sensitive to the level of the rate. By χ^2 tests to US T-bill data, the above models which assume $\theta < 1$ (or $\beta < 1$) are rejected and those which assume $\theta \geq 1$ (or $\beta \geq 1$) are not rejected. For example, applying the Generalized Method Moment, Chan et al. [11] give $\theta = 1.449$. Using the same data, by the Gaussian Estimation methods, Nowman [12] estimates $\theta = 1.361$. Therefore, it is more evident to consider the SDEs with $\theta \geq 1$ and $\beta \geq 1$. However there is so far no result on the numerical solutions for the SDE model (1.6) when $\theta, \beta > 1$. The aim of this paper is to close this gap. We will show that the EM numerical solutions will converge to the true solution in probability. We will also show that the EM solutions can be used to compute some financial quantities of the SDE model including the option value, for example.

It is essential for the SDE model (1.6) to have its non-negative solution. Given that the SDE does not obey the linear growth condition though it satisfies the local Lipschitz condition, there is so far no result on the non-negative solution. We will therefore in Section 2 develop a technique to prove the non-negativity of the solution to the model. In Section 3, we will define the EM approximate solutions to the volatility process $V(t)$ and the underlying asset price process $X(t)$. To guarantee the non-negativity of the EM solutions, we will use the technique of stopping times. We will show that the EM numerical solutions will converge to the true solution in probability. To demonstrate the practical use of the EM numerical method, we will show in Section 4 that the EM solutions can be used to compute several important financial quantities of the SDE model including the option value.

1.1. Notation

Throughout this paper, unless otherwise specified, we will use the following notation. Let $\big(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ be a complete probability space with filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathfrak{F}_0$ contains all $\mathbb{P}$-null sets). Let w_1 and w_2 be scalar Brownian motions defined on the probability space and w_1 and w_2 have their correlation coefficient ρ. For a pair of real numbers a and b, we let $a \wedge b = \min\{a, b\}$. For a set A, denote its indicator function by 1_A. We also set $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). Moreover, we let T be an arbitrary positive number.

2. The non-negative solution

The SDE model (1.6) describes the asset price and its volatility in the financial market. It is therefore essential to prove that the solution of (1.6) is non-negative with probability 1. The following lemmas in fact show that the solution is positive with probability 1.

2.1. Non-negative $V(t)$

Lemma 2.1. *Let $\beta > 1$. Then, for any given initial value $V(0) = V_0 > 0$, the solution $V(t)$ of the SDE model (1.6) will be positive for all $t \in [0, T]$ almost surely.*

Proof. Treat the second SDE in (1.6) as an SDE in the whole real space $\mathbb{R} = (-\infty, \infty)$ by setting its coefficients to be 0 when $V(t) < 0$. Clearly, the coefficients obey the local Lipschitz condition. Hence, there exists a unique maximal local solution $V(t)$ on $t \in [0, \rho_e)$, where ρ_e is the explosion time (see e.g. [20]). For any sufficiently large positive number M, namely $\frac{1}{M} < V(0) < M$, define a stopping time $\rho_M = \rho_e \wedge \inf\big\{t \in [0, \rho_e) : |V(t)| \notin [\frac{1}{M}, M]\big\}$ and set $\rho_\infty = \lim_{M\to\infty} \rho_M$.

Now, define a C^2-function $H : (0, \infty) \to (0, \infty)$ by

$$H(V) = V^{\frac{1}{2}} - 1 - \frac{1}{2}\ln V, \quad V > 0.$$