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On difference equations with asymptotically stable 2-cycles perturbed by a decaying noise[☆,☆☆]

E. Braverman[a], A. Rodkina[b,*]

[a] Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N. W., Calgary, AB T2N 1N4, Canada

[b] Department of Mathematics, The University of the West Indies, Mona Campus, Kingston, Jamaica

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ABSTRACT

The results stating that the stability of a 2-cycle is preserved (almost surely) under an eventually decaying stochastic perturbation are obtained in the case when the system is in the range of the parameters immediately following the Hopf bifurcation and preceding the next period doubling bifurcation. Several examples of systems and types of noise are presented.

1. Introduction

Difference equations are applied as adequate models of population dynamics, for example, for species with seasonal reproduction. There are many publications concerning stability of discrete nonlinear systems, preservation of stability under deterministic and stochastic perturbations, see, for example, recent publications [1–5] and references therein. However, oscillatory (and in particular 2-cyclic) behavior is characteristic for many real world systems, and is more frequently observed than convergence to a stable equilibrium, and sustainable oscillations cannot be explained by random noise only. Meanwhile, there are very few papers where the research is focused on stable oscillatory behavior and in particular on stabilization by a random perturbation or noise [6–11]. Also, if the cyclic behavior of the nonperturbed equation is combined with a stochastic perturbation, such systems have so far received very little attention in the literature. In [6] it is explained how random perturbations can cause blurred stable orbits in an otherwise chaotic systems, see also [7].

In the present paper we assume that the original system is in the range of parameters leading to a stable 2-cycle and deduce conditions under which the orbits of a stochastically perturbed system eventually stay in a δ-neighborhood of this 2-cycle with a probability $1-\gamma$, for any given small $\delta > 0$, $\gamma > 0$. Stochastic perturbations of stable 2-cycles are studied in the case when the range of the parameters is between the first and the second period doubling bifurcations. The well-known logistic $F_\mu(x) = \mu x(1-x)$, Ricker $F_\mu(x) = x\exp(\mu(1-x))$, Hassel and May $F_\mu(x) = \mu x(1+x)^{-d}$, $d > 1$, and Bellows maps $F_\mu(x) = \mu x(1+(ax)^d)^{-1}$, $a > 0,\ d > 1$, for appropriate values of the parameter $\mu > 0$, are examples of such systems (see more details in Section 5.1).

In this paper a scalar difference equation depending on parameter is perturbed by a vanishing stochastic noise ρ_n such that, almost surely,

$$\rho_n \to 0 \quad \text{as } n \to \infty. \tag{1}$$

The paper is organized as follows. After some preliminaries in Section 2, a scalar difference equation with an asymptotically stable 2-cycle perturbed by a decaying noise is considered in Section 3. The main result is Theorem 3.1 which claims that if

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* Corresponding author.
E-mail addresses: maelena@math.ucalgary.ca (E. Braverman), alexandra.rodkina@uwimona.edu.jm, alechkajm@yahoo.com (A. Rodkina).

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the equation has a stable 2-cycle, then under a vanishing noise positive solutions tend to this cycle almost surely. In Section 4 we discuss how to find $j(\gamma)$ satisfying

$$\mathbb{P}\{|\rho_n| < j, \text{ for all } n \in \mathbf{N}\} > 1 - \gamma,$$

when condition (1) holds, $\rho_n = \sigma_{n-1}\xi_n$, where σ_{n-1} are nonrandom coefficients and ξ_n are random variables. We compute the values of $j(\gamma)$ for two types of σ_n and for $\gamma = 0.05$, when distributions of ξ_n have square-exponential tails. In Section 5 we describe some well-known maps along with the range of parameters which provide an asymptotically stable limit cycle. In conclusion we present a numerical example of the logistic equation perturbed by a decaying noise.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. A detailed discussion of probabilistic concepts and notation may be found, for example, in [12].

Everywhere in this paper we suppose that assumption (1) holds a.s. Conditions which guarantee (1) are given in [2] (see also [4]). The next lemma is proved in [2].

Lemma 2.1. *Let (1) holds a.s. Then $\forall\gamma \in (0, 1)$ there exist $\Omega_\gamma \subseteq \Omega$ and $j(\gamma)$ such that*

$$\sup_{n\in\mathbf{N}} |\rho_n(\omega)| < j(\gamma), \quad \omega \in \Omega_\gamma, \ \mathbb{P}(\Omega_\gamma) > 1 - \gamma. \tag{2}$$

In the proof of the main theorem we will use the following elementary lemma, whose proof we present for completeness of the argument.

Lemma 2.2. *Let $k_1 \in (0, 1)$, $\beta_i \geq 0$ for each $i \in \mathbf{N}$, and $\lim_{i\to\infty} \beta_i = 0$. Then*

$$\sum_{i=0}^{n} k_1^i \beta_{n-i} \to 0, \quad \text{as } n \to \infty.$$

Proof. Fix $\varepsilon_0 > 0$. Let $N_1 \in \mathbf{N}$ be so large that

$$\frac{\beta k_1^{N_1+1}}{(1 - k_1)} < \varepsilon_0/2.$$

Suppose that $N_2 \in \mathbf{N}$ is so large that for $n \geq N_2$

$$\beta_n < \frac{\varepsilon_0(1 - k_1)}{2}.$$

Then, for $n > N_2 + N_1$,

$$\begin{aligned}
\sum_{i=0}^{n} k_1^i \beta_{n-i} &= \sum_{i=0}^{N_1} k_1^i \beta_{n-i} + \sum_{i=N_1+1}^{n} k_1^i \beta_{n-i} \\
&\leq \max_{j\geq N_2}\{\beta_j\} \sum_{i=0}^{N_1} k_1^i + \beta k_1^{N_1+1} \sum_{i=0}^{\infty} k_1^i \\
&\leq \frac{\varepsilon_0(1 - k_1)}{2} \frac{1}{1 - k_1} + \beta k_1^{N_1+1} \frac{1}{1 - k_1} < \varepsilon_0. \quad \square
\end{aligned}$$

3. Main result

Let $F_\mu : \mathbf{R} \to \mathbf{R}$ be continuously differentiable for $\mu \in (\bar{\mu}, \hat{\mu})$. We define

$$F_\mu^2(x) := F_\mu(F_\mu(x)), \tag{3}$$

and note that F_μ^2 is also continuously differentiable on $\mathbf{R}$.

Suppose that the equation

$$x_{n+1} = F_\mu(x_n), \quad n \in \mathbf{N}, \tag{4}$$

has a 2-cycle, $\{x_\mu(0), x_\mu(1)\}$, when $\mu \in (\mu_1, \mu_2) \subset (\bar{\mu}, \hat{\mu})$. Moreover, suppose that for each $\mu \in (\mu_1, \mu_2)$

$$\left|(F_\mu^2)'(x_\mu(0))\right| = \left|F_\mu'(x_\mu(0))\right| \left|F_\mu'(x_\mu(1))\right| = k_\mu < 1. \tag{5}$$

Condition (5) implies that a 2-cycle, $\{x_\mu(0), x_\mu(1)\}$, is asymptotically stable for each $\mu \in (\mu_1, \mu_2)$ (see [13, Theorem 1.22, page 39]).