Computers and Mathematics with Applications 64 (2012) 2262–2274

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Sharp algebraic periodicity conditions for linear higher order difference equations

István Győri, László Horváth *

Department of Mathematics, University of Pannonia, 8200 Veszprém, Egyetem u. 10., Hungary

ARTICLE INFO

Keywords:
Periodic solution
Higher order difference equation
Companion matrix

ABSTRACT

In this paper we give easily verifiable, but sharp (in most cases necessary and sufficient) algebraic conditions for the solutions of systems of higher order linear difference equations to be periodic. The main tool in our investigation is a transformation, recently introduced by the authors, which formulates a given higher order recursion as a first order difference equation in the phase space. The periodicity conditions are formulated in terms of the so-called companion matrices and the coefficients of the given higher order equation, as well.

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1. Introduction

In this paper we derive new necessary and sufficient, and sufficient algebraic conditions on the periodicity of the solutions of the d-dimensional system of the sth order difference equations

$$x(n) = \sum_{i=1}^{s} A_i(n)x(n-i), \quad n \geq 0, \tag{1}$$

where

(C_1) $s \geq 1$ is a given integer, and $A_i(n) \in \mathbb{R}^{d\times d}$ for every $1 \leq i \leq s$ and $n \geq 0$.

It is clear that the solutions of (1) are uniquely determined by their initial values

$$x(n) = \varphi(n), \quad -s \leq n \leq -1, \tag{2}$$

where $\varphi(n) \in \mathbb{R}^d$. The unique solution of (1) and (2) is denoted by $x(\varphi) = (x(\varphi)(n))_{n\geq -s}$, where the block vector $\varphi := (\varphi(-s), \dots, \varphi(-1))^T \in V_s$. Here V_s means the sd-dimensional real vector space of block vectors with entries in $\mathbb{R}^d$.

We believe that our results about Eq. (1) are interesting in their own right. Further, we believe that these results offer prototypes toward the development of the theory of the periodic behavior of the solutions of nonlinear higher order difference equations.

This equation and its special cases are studied in many textbooks on difference equations such as [1–6], and so on.

On p. 25 in the book [3], Grove and Ladas put the following two questions:

"What is it that makes every solution of a difference equation periodic with the same period?"

"Is there any easily verifiable test that we can apply to determine whether or not this is true?"

Motivated by the above questions, and the papers [7,8], we worked out an easily verifiable algebraic test that we can apply to determine the p-periodic solutions of a linear higher order difference equation. In our results we obtain precise analysis of the periodicity of the solutions not only for the scalar but for the vector case. Note that for this latter case, to the best knowledge of the authors, there are no similar results in the literature.

* Corresponding author.
E-mail addresses: gyori@almos.uni-pannon.hu (I. Győri), lhorvath@almos.uni-pannon.hu, lhorvath@almos.vein.hu (L. Horváth).

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doi:10.1016/j.camwa.2012.02.018

As an illustration, we formulate two typical applications of our main results. Consider the special case of (1)

$$x(n) = \sum_{i=1}^{s} A_i x(n-i), \quad n \geq 0, \tag{3}$$

where $A_i \in \mathbb{R}^{d\times d}$ for every $1 \leq i \leq s$.

For the integers $1 \leq p \leq s$, V_s^p denotes the set of all initial vectors $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$ such that $\varphi(i) = \varphi(j)$ if $i \equiv j(\bmod p)(i, j = -s, \ldots, -1)$. Of course $V_s^s = V_s$.

If $a \in \mathbb{R}$, then $[a]$ denotes the greatest integer that does not exceed a.

Theorem 1.1. *Suppose $1 \leq p \leq s$ is an integer, let $u := \left[\frac{s}{p}\right]$, and let $v := s - up (0 \leq v \leq p-1)$. Then for every $\varphi \in V_s^p$ the solution of (3) and (2) is p-periodic if and only if one of the following conditions holds:*

(a_1) $v = 0$ *and*

$$\sum_{j=0}^{u-1} A_{jp+i} = O, \quad 1 \leq i < p; \qquad \sum_{j=0}^{u-1} A_{jp+p} = I. \tag{4}$$

(a_2) $0 < v < p-1$, *moreover*

$$\sum_{j=0}^{u} A_{jp+i} = O, \quad 1 \leq i \leq v; \qquad \sum_{j=0}^{u-1} A_{jp+i} = O, \quad v+1 \leq i < p,$$

and

$$\sum_{j=0}^{u-1} A_{jp+p} = I.$$

(a_3) $v = p-1$ *and*

$$\sum_{j=0}^{u} A_{jp+i} = O, \quad 1 \leq i < p; \qquad \sum_{j=0}^{u-1} A_{jp+p} = I.$$

Remark 1.1. If $p = s$, then $v = 0$ and (4) can be written in the form

$$A_i = O, \quad 1 \leq i \leq s-1, \qquad A_s = I.$$

Theorem 1.2. *Suppose $p > s$ is an integer. Then for every $\varphi \in V_s$ the solution of (3) and (2) is p-periodic if and only if the rank of the matrix*

$$\begin{pmatrix} B_p - I & B_{p-1} & \ldots & B_2 & B_1 \\ B_1 & B_p - I & \ldots & B_3 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p-1} & B_{p-2} & \ldots & B_1 & B_p - I \end{pmatrix} \tag{5}$$

with entries

$$B_i := A_i, \quad 1 \leq i \leq s \quad \text{and} \quad B_i := O, \quad s+1 \leq i \leq p$$

is equal to $d(p-s)$.

Theorems 1.1 and 1.2 are consequences of our Theorems 4.1–4.3. For the definition of the periodic solutions see Section 2 below.

It is known that the system (1) can be reformulated to a ds-dimensional system of first order difference equations in an appropriate sequence space (see e.g. [2]). The matrices of the first order ds-dimensional system are called companion matrices of Eq. (1) and the system itself is called a companion system of (1). It is clear that the companion matrices are block matrices defined by all $A_i(n)$ in Eq. (1). Recently, Győri and Horváth introduced a new transformation which is extremely useful in analyzing the summability of the solutions of higher order difference equations (see e.g. [9,10]). In this paper we show that our transformation is also powerful in studying the periodicity of the solutions of the Eq. (1).

Our paper is essentially subdivided into six parts.

Section 2 is fundamental for our work and contains basic results on our transformation of the Eq. (1) into a first order system with tractable companion matrices.

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